Math 2280-1
Tuesday March 15

§5.5 Matrix exponentials & linear systems of DE's
(first finish the damped oscillating train example from §5.4)

Def: Consider

$*\quad \vec{x}\,' = A\vec{x}$

$A_{n\times n}$ constant matrix

If $\{\vec{x}_1(t), \vec{x}_2(t), \ldots \vec{x}_n(t)\}$ is a basis of sol'ns to $*$

then

$$\Phi(t) = \left[\begin{array}{c|c|c} \vec{x}_1(t) & \cdots & \vec{x}_n(t) \end{array}\right]$$

is called a $\left.\begin{array}{c}\underline{\text{fundamental}}\\ \underline{\text{matrix}}\\ \underline{\text{solution}}\end{array}\right\}$ FMS

Notice, this is equivalent to saying $X(t) = \Phi(t)$ solves

(just look column by column!)

$$\begin{cases} X' = A\,X_{n\times n} \\ X(0) \quad \text{non-singular (i.e. invertible, rank } n, \det \neq 0) \end{cases}$$

Example 1 p.350

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 10 = (\lambda-5)(\lambda+2)$$

$\lambda = 5$

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 3 & -6 & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda = -2$

$$\left[\begin{array}{cc|c} 6 & 2 & 0 \\ 3 & 1 & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$$\vec{x}_1(t) = e^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \vec{x}_2(t) = e^{-2t}\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

possible $\Phi(t) = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix}$

<u>Theorem</u> If $\Phi(t)$ is a FMS for * then the (unique) sol'n to

$$\text{IVP} \quad \begin{cases} \vec{x}' = A\vec{x} \\ \vec{x}(0) = \vec{x}_0 \end{cases}$$

is $\vec{x}(t) = \Phi(t)\left[\Phi^{-1}(0)\,\vec{x}_0\right]$

pf: the general sol'n to * is

$$\vec{x}_H(t) = \Phi(t)\,\vec{c} \quad \left(= c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n\right)$$

so the $\vec{x}(t) = \Phi(t)\left[\Phi^{-1}(0)\vec{x}_0\right]$ solves *

and, $\vec{x}(0) = \left[\Phi(0)\Phi^{-1}(0)\right]\vec{x}_0 = I\,\vec{x}_0 = \vec{x}_0$

so $\vec{x}(t)$ solves IVP ∎

<u>Example 1 cont'd</u>

solve

$$\begin{cases} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{cases}$$

using page 1 $\Phi(t)$ and theorem above.

ans: $\vec{x}(t) = \frac{3}{7}e^{-2t}\begin{bmatrix} 1 \\ -3 \end{bmatrix} + \frac{2}{7}e^{5t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Remark: If $\Phi(t)$ is a FMS and C is invertible,

then so is $\Phi(t)C$ (note - you multiply by C on the right)

since

$$\frac{d}{dt}\left(\Phi(t)\,C\right) = \Phi'(t)C \qquad \leftarrow \text{universal product rule, see page 6}$$

$$= (A\Phi)C$$

$$= A(\Phi C)$$

and $\Phi(0)C$ is invertible.

$\Phi(t)\Phi^{-1}(0)$ is the best FMS because it is the unique soltn to

$$\begin{cases} \dfrac{dX}{dt} = AX \\ X(0) = I \end{cases}$$

because $col_j(X)$ is the unique soltn to

$$\begin{cases} \dfrac{d\vec{x}}{dt} = A\vec{x} \\ \vec{x}(0) = \vec{e}_j \end{cases}$$

It's so special we will call it e^{At}!

Notice, the sol'n to the IVP

$$\begin{cases} \dfrac{d\vec{x}}{dt} = A\vec{x} \\ \vec{x}(0) = \vec{c} \end{cases}$$

is $e^{At}\vec{c}$, "just" like for the Chapter 1 scalar eqtn.

Example 1 cont'd

$$e^{At} = \Phi(t)\Phi(0)^{-1} = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \frac{1}{7} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} e^{-2t} + 6e^{5t} & -2e^{-2t} + 2e^{5t} \\ -3e^{-2t} + 3e^{5t} & 6e^{-2t} + e^{5t} \end{bmatrix}$$

But wait!

Didn't you like how we derived Euler's formula?

Here's a different (?) way to define e^{At}:

$$A_{n\times n} \qquad e^{A} := I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots + \frac{1}{k!}A^k + \dots$$

(1220: notice this converges for any $A_{n\times n}$, since if ~~the biggest any~~ $|a_{ij}| \le M \ \forall ij$

then $|\text{entry}_{ij}(A^2)| \le nM^2$

$|\text{entry}_{ij}(A^3)| \le n^2M^3$

$|\text{entry}_{ij}(A^k)| \le n^{k-1}M^k$

so the ij entry series converges absolutely (dominated by the series for e^{Mn})

$$\boxed{e^{tA} := I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \dots + \frac{t^k}{k!}A^k + \dots}$$

notice, for $X(t) := e^{tA}$

$$\begin{cases} X(0) = I \\ X'(t) = A + \frac{2t}{2!}A^2 + \frac{t^2}{2!}A^3 + \dots + \frac{t^{k-1}}{(k-1)!}A^k + \dots \\ \quad = A\left(I + tA + \frac{t^2}{2!}A^2 + \dots\right) \\ \quad = AX \end{cases}$$

(Maybe in 3220 you'll learn why you can diff this ∞ series term by term)

Theorem

Since the power series def of e^{tA} solves $\begin{cases} X' = AX \\ X(0) = I \end{cases}$ (and sol'n is unique) because each column is unique sol'n to a specific IVP

It must be that

$$\boxed{e^{tA} = \Phi(t)\Phi^{-1}(0) = I + tA + \frac{t^2}{2!}A^2 + \dots + \frac{t^k}{k!}A^k + \dots}$$

Example 1 cont'd (2270!!)

$$A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

$$S = \left[\vec{v}_1 \middle| \vec{v}_2 \right] = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \qquad AS = \left[-2\vec{v}_1 \middle| 5\vec{v}_2 \right] = \left[\vec{v}_1 \middle| \vec{v}_2 \right] \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} = S\Lambda$$

$$\Lambda = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} = S^{-1}AS$$

so
$$A = S\Lambda S^{-1}$$
$$A^2 = S\Lambda S^{-1} S \Lambda S^{-1} = S\Lambda^2 S^{-1}$$
$$A^k = S\Lambda^k S^{-1} \longleftarrow \text{notice } \Lambda^k = \begin{bmatrix} (-2)^k & 0 \\ 0 & 5^k \end{bmatrix} \text{ is easy to compute!}$$

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \dots + \frac{t^k}{k!}A^k + \dots$$
$$= I + tS\Lambda S^{-1} + \frac{t^2}{2!}S\Lambda^2 S^{-1} + \dots + \frac{t^k}{k!}S\Lambda^k S^{-1}$$
$$= S\left[I + t\Lambda + \frac{t^2}{2!}\Lambda^2 + \dots + \frac{t^k}{k!}\Lambda^k + \dots \right] S^{-1}$$
$$= S \begin{bmatrix} \sum_{k=0}^{\infty} \frac{(-2t)^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{(5t)^k}{k!} \end{bmatrix} S^{-1}$$
$$= \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{5t} \end{bmatrix} \frac{1}{7} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$
$$= \frac{1}{7} \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & -2e^{-2t} \\ 3e^{5t} & e^{5t} \end{bmatrix}$$
$$= \frac{1}{7} \begin{bmatrix} e^{-2t} + 6e^{5t} & -2e^{-2t} + 2e^{5t} \\ -3e^{-2t} + 3e^{5t} & 6e^{-2t} + e^{5t} \end{bmatrix}$$

same as page 3!!!

Universal product rule

Recall the 1-var. proof of the product rule, write $*$ for "times", and see the generality in which it really applies: (t is a real variable)

$$D_t(f*g)(t) = \lim_{\Delta t\to 0}\left[f(t+\Delta t)*g(t+\Delta t) - f(t)*g(t)\right]\frac{1}{\Delta t}$$

$\boxed{\text{Def. of deriv}}$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\left[f(t+\Delta t)-f(t)\right]*g(t+\Delta t) + f(t)*\left[g(t+\Delta t)-g(t)\right]\right]$$

$\boxed{\text{mult } * \text{ distributes over addition on both sides}}$

$$= \lim_{\Delta t\to 0}\left[\frac{1}{\Delta t}\left(f(t+\Delta t)-f(t)\right)\right]*g(t+\Delta t) + \lim_{\Delta t\to 0} f(t)*\left[\frac{1}{\Delta t}\left(g(t+\Delta t)-g(t)\right)\right]$$

$\boxed{\text{Scalar mult distributes into either factor of the product}}$

$$\boxed{D_t(f*g)(t) = f'(t)*g(t) + f(t)*g(t)}$$

$\boxed{\text{limit of product is product of limits. \& differentiable fcns are continuous}}$

<u>examples</u>
- function times fcn (1210)
- dot product, cross product } 2210
- scalar fcn times vector fcn, matrix fcn times vector fcn, matrix fcn times matrix fcn } 2280

<u>not an example</u> composition, $f\circ g$!
(which is why we have the chain rule.)