Math 2280-1
Friday March 11

Experiments First! p. 6-7

(and Monday March 14)

Hw for 3/19 (1)

- Maple exploration §5.3
  Understand set up, then do the project work in the paragraph at bottom of page 331; "For your personal seven-story ..."
  You don't need to hand in the warm-up problems.
- 5.4 1,7,11,29 Work 1,7,11 by hand, except for computer to draw #1 phase portrait. You may work 29 by hand, but I'd recommend Maple to help you find chains.
- 5.5 1,3,11,23,36 Work by hand, except Maple for eigenvectors. Also have Maple check exp(At), and hand in

§ 5.4 : $1^{st}$ order systems $\frac{d\vec{x}}{dt} = A\vec{x}$

when $A$ is not diagonalizable

Recall (2270),
for any $A_{n\times n}$,

$$|A-\lambda I| = (-1)^n(\lambda-\lambda_1)^{k_1}(\lambda-\lambda_2)^{k_2}\cdots(\lambda-\lambda_m)^{k_m}$$

$\lambda_i \in \mathbb{C}$ $\quad 1 \le i \le m$
$\lambda_i$ distinct

$$k_1 + k_2 \cdots + k_m = n$$

$E_{\lambda_i} = \lambda_i$-eigenspace

- $\dim(E_{\lambda_i}) \le k_i$ always holds (geometric multiplicity $\le$ algebraic mult.)
- $A$ is diagonalizable (has an $\mathbb{R}^n$ eigenbasis) iff each $\dim(E_{\lambda_i}) = k_i$

and in this (diagonalizable) case, we can find the general soltn to $\frac{d\vec{x}}{dt} = A\vec{x}$ as linear combos of $\{e^{\lambda_1 t}\vec{v}_1, \ldots e^{\lambda_m t}\vec{v}_n\}$.

($\exists$ analog for complex evals & evects)

example 1 p 333

$$\vec{x}\,' = \begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix}\vec{x}$$

$$|A-\lambda I| = -(\lambda-5)(\lambda-3)^2$$

$\lambda = 5$

$$\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda = 3$:

$$\left[\begin{array}{ccc|c} 6 & 4 & 0 & 0 \\ -6 & -4 & 0 & 0 \\ 6 & 4 & 0 & 0 \\ \hline 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

---

$x_3 = t$
$x_2 = s$
$x_1 = -\frac{2}{3}s$

$$\vec{x} = s\begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\{\vec{v}_1, \vec{v}_2\} = \left\{\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$$\vec{x}_H(t) = c_1 e^{5t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + c_3 e^{3t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

all is well!
(even though $\lambda = 3$ had algebraic mult $= 2$).

But, how about

$$\vec{x}' = \underbrace{\begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -9 & 9 \end{bmatrix}}_{B} \vec{x}$$

$$|B-\lambda I| = \begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & -3-\lambda & 4 \\ 0 & -9 & 9-\lambda \end{vmatrix} = (5-\lambda)\left[\lambda^2 - 6\lambda \overbrace{-27+36}^{+9}\right]$$
$$= (5-\lambda)(\lambda-3)^2 \qquad \text{same as example 1.}$$

$\lambda = 5$ $\quad \vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\qquad \lambda = 3$ $\quad \vec{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ (Maple)

$\lambda = 3$ eigenspace is "defective"

$$\left\{ e^{5t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e^{3t}\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, ?? \right\}$$

discussion:

Let $A\vec{v} = \lambda\vec{v}$, but $\lambda$-eigenspace "defective".

$e^{\lambda t}\vec{v}$ solves $\frac{d\vec{x}}{dt} = A\vec{v}$

does $\vec{z}(t) = te^{\lambda t}\vec{v}$ ?

$$\frac{d\vec{z}}{dt} = e^{\lambda t}\vec{v}\,[1+\lambda t]$$
$$A\vec{z} = te^{\lambda t}A\vec{v} = te^{\lambda t}\lambda\vec{v}$$

$$\frac{d\vec{z}}{dt} - A\vec{z} = e^{\lambda t}\vec{v} \qquad \text{so } \vec{z}(t) \text{ fails.}$$

how about

$$\vec{z}(t) = e^{\lambda t}(t\vec{v} + \vec{w}) \ ?$$
$$\vec{z}'(t) = e^{\lambda t}[\lambda t\vec{v} + \lambda\vec{w} + \vec{v}]$$
$$A\vec{z} = e^{\lambda t}[t\lambda\vec{v} + A\vec{w}]$$

success iff $\boxed{A\vec{w} = \lambda\vec{w} + \vec{v}}$

$\boxed{(A-\lambda I)\vec{w} = \vec{v}}$

in our example: $\left(\lambda = 3, \vec{v} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}\right)$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & -6 & 4 & 2 \\ 0 & -9 & 6 & 3 \end{array}\right]$$

$R_2/{-2}$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 3 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$w_3 = t$
$w_2 = -\frac{1}{3} + \frac{2}{3}t$
$w_1 = 0$

$$\vec{w} = \begin{bmatrix} 0 \\ -1/3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 2/3 \\ 1 \end{bmatrix}$$

mult of $\vec{v}$

can take $\vec{w} = \begin{bmatrix} 0 \\ -1/3 \\ 0 \end{bmatrix}$

get soltn

$$?? = \vec{z}(t) = e^{\lambda t}\left\{ t\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/3 \\ 0 \end{bmatrix} \right\}$$

this gives a 3rd lin ind sol'n!!

Theorems : Let $A_{n\times n}$ as on page 1, with

$$|A-\lambda I| = (-1)^n (\lambda-\lambda_1)^{k_1}(\lambda-\lambda_2)^{k_2}\cdots(\lambda-\lambda_m)^{k_m}$$

$k_1+\dots+k_m = n$
$\lambda_i \in \mathbb{C} \quad 1\le i\le m$
$\lambda_i$ distinct

(1) The generalized $\lambda_i$-eigenspace is defined to be

$$\ker (A-\lambda_i I)^{k_i}$$

it is always $k_i$ dimensional (in $\mathbb{C}^n$)

(2) By amalgamating bases of the generalized eigenspaces you get a basis for $\mathbb{C}^n$

(3) The generalized $\lambda_i$-eigenbases can be chosen to be made of one or more "chains" $\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_k\}$ where for $\lambda = \lambda_i$

$$\begin{aligned}(A-\lambda I)\vec{u}_1 &= \vec{0} \\ (A-\lambda I)\vec{u}_2 &= \vec{u}_1 \\ &\vdots \\ (A-\lambda I)\vec{u}_k &= \vec{u}_{k-1}\end{aligned}$$

($\vec{u}_1$ is an eigenvector)

( on page 2, $(\vec{v},\vec{w})$ was a chain of length 2 )

(4) Each chain $\{\vec{u}_1, \vec{u}_2, \dots \vec{u}_k\}$ yields $k$ lin. ind. sol'ns to $\frac{d\vec{x}}{dt} = A\vec{x}$ :

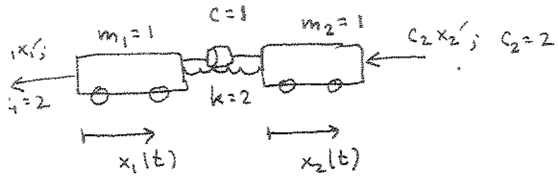
$$\begin{aligned}\vec{x}_1(t) &= \vec{u}_1 e^{\lambda t} \\ \vec{x}_2(t) &= (\vec{u}_1 t + \vec{u}_2)e^{\lambda t} \\ \vec{x}_3(t) &= (\vec{u}_1 \tfrac{1}{2}t^2 + \vec{u}_2 t + \vec{u}_3)e^{\lambda t} \\ &\vdots \\ \vec{x}_k(t) &= (\vec{u}_1 \tfrac{1}{k!}t^k + \dots + \vec{u}_{k-1}t + \vec{u}_k)e^{\lambda t}\end{aligned}$$

$\leftarrow$ notice $\vec{x}_j{}' = \lambda\vec{x}_j + \vec{x}_{j-1}$
$A\vec{x}_j = \lambda\vec{x}_j + \vec{x}_{j-1}$ too!

amalgamating these solutions yields a basis of sol'ns to $\frac{d\vec{x}}{dt} = A\vec{x}$ !!!

proof we will discuss aspects of the general proof during following lectures
↑
(might)

## Example 6 p. 343




$$m_1 x_1'' = +k(x_2 - x_1) - c_1 x_1' + c(x_2' - x_1')$$
$$m_2 x_2'' = -k(x_2 - x_1) - c_2 x_2' - c(x_2' - x_1')$$

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

cannot use § 5.3 (no damping.)
Must convert to 1st order system

$$\begin{aligned} x_1 &= x_1 \\ x_2 &= x_2 \\ x_3 &= x_1' \\ x_4 &= x_2' \end{aligned} \qquad \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Maple to the rescue! (See next page).

$$\begin{bmatrix} x_1 \\ x_2 \\ x_1' \\ x_2' \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + e^{-2t}\left[ c_2 \begin{bmatrix} 1 \\ 1 \\ -2 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix} + c_4 \overbrace{\left[ t \begin{bmatrix} -1 \\ 1 \\ 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right]}^{t\vec{u}_1 + \vec{u}_2} \right]$$

explain.

nice physically; the sum of MAPLE's 2 $\lambda=-2$ eigenvects, times −2.

the difference of Maple's 2 $\lambda=-2$ eigenvects, ×2. also nice physical interpretation defective eigenspace computations

```
> with(LinearAlgebra):
> A := Matrix(4, 4, [0, 0, 1, 0, 0, 0, 0, 1,
        -2, 2,-3, 1, 2,-2, 1,-3]);
```

$$A := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{bmatrix} \quad (1)$$

```
> Eigenvectors(A);
```

$$\begin{bmatrix} 0 \\ -2 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (2)$$

```
> Iden := Matrix(4, shape = identity);
```

$$Iden := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3)$$

```
> B := A + 2·Iden; #eigenvalue of A is −2, this is A−λ·I
```

$$B := \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -1 \end{bmatrix} \quad (4)$$

```
> NullSpace(B^3); # the nullspace of (A−λ·I)^k is k − dimensional, where k is the algebraic multiplicity of λ
  NullSpace(B^2);
  NullSpace(B);
```

$$\left\{ \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\} \quad (5)$$

```
>
```

From the above we deduce that the chain structure of the generalized $\lambda = -2$ nullspace is one chain of length two and one chain of length 1.

```
> u2 := Vector([0, 0,-1, 1]);
      #for chain of llength 2 don't start with an eigenvector.  I like this one for its physical meaning −
    #it's the sum of the first two generalized eigenvectors.
  u1 := B.u2;
```

$$u2 := \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$u1 := \begin{bmatrix} -1 \\ 1 \\ 2 \\ -2 \end{bmatrix} \quad (6)$$

# Calculations for a 2 mass– 3 spring system

Math 2280–1

March 11, 2011

**The two mass, three spring system.**

Data: Each mass is 50 grams. Each spring mass is 10 grams. (Remember, and this is a defect, our model assumes massless springs.) The springs are "identical", and a mass of 50 grams stretches the spring 15.6 centimeters. (We can recheck this.). Thus the spring constant is given by

```
> Digits := 4 :
> solve(k*.156=.05*9.806,k);
```

$$3.143 \tag{1}$$

Let's time the two natural periods (which we discuss below):

Here's the model:

```
> with(LinearAlgebra):
> A:=Matrix(2,2,[-2*k/m, k/m,k/m,-2*k/m]);
    #this should be the "A" matrix you get for
    #our two-mass, three-spring system.
```

$$A := \begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{bmatrix} \tag{2}$$

```
> Eigenvectors(A);
```

$$\begin{bmatrix} -\frac{3k}{m} \\ -\frac{k}{m} \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \tag{3}$$

Predict the two natural periods from the model:

---

**ANSWER:** If you do the model correctly and my office data is close to our class data, you will come up with theoretical natural periods of close to .46 and .79 seconds. I predict that the actual natural periods are a little longer, especially for the slow mode. (In my office experiment I got periods of 0.482 and 0.855 seconds.) What happened?

**EXPLANATION:** The springs actually have mass, equal to 10 grams each. This is almost on the same order of magnitude as the yellow masses, and causes the actual experiment to run more slowly than our model predicts. In order to be more accurate the total energy of our model must account for the kinetic energy of the springs. You actually have the tools to model this more–complicated situation, using the ideas of total energy discussed in section 5.6, and a "little" Calculus. You can carry out this analysis, like I sketched for the single mass, single spring oscillator (feb11.pdf), assuming that the spring velocity at a point on the spring linearly interpolates the velocity of the wall and mass (or mass and mass) which bounds it. It turns out that this gives an A–matrix the same eigenvectors, but different eigenvalues, namely

$$\lambda_1 = -\frac{6\,k}{6\,m + 5\,m_s}$$

$$\lambda_2 = -\frac{6\,k}{2\,m + m_s}.$$

(Hints: the "M" matrix is not diagonal but the "K" matrix is the same.)

If you use these values, then you get period predictions

```
> m:=.050;
  ms:=.010;
  k:=3.143;
  Omega1:=sqrt(6*k/(6*m+5*ms));
  Omega2:=sqrt(6*k/(2*m+ms));
  T1:=evalf(2*Pi/Omega1);
  T2:=evalf(2*Pi/Omega2);
```

$$m := 0.050$$

$$ms := 0.010$$

$$k := 3.143$$

$$\Omega 1 := 7.340$$

$$\Omega 2 := 13.09$$

$$T1 := 0.8559$$

$$T2 := 0.4801 \qquad (4)$$

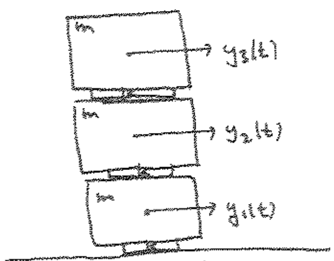
of .856 and .480 seconds per cycle. Is that closer?

**Challenge: If you can construct (and explain to me in my office) a correct derivation of the eigenvalues /eigenvectors I claim above, by taking the spring masses into account, then you can either substitute your derivation for the section 5.3 Maple exploration in this week's homework, or get 10 bonus points on the next midterm. This is a challenging challenge, but it's definitely doable!**

## Earthquake Project Comments

(A careful explanation of the book's claim that even though the building is only being shaken at the base, in a frame of reference moving with ground, each floor experiences an equal "inertial force".)

consider a 3 story building (in project is 7 stories) modeled as a mass-spring model, with each story having mass m, and floor junctions modeled as "springs" with constants k (with respect to transverse as opposed to parallel motion)




→ transverse (shaking) wave $E\cos\omega t$

$y_0(t) = E\cos\omega t$ so

$$y_0'' = -E\omega^2\cos\omega t$$
$$m y_1'' = -k(y_1 - y_0) - k(y_1 - y_2)$$
$$m y_2'' = -k(y_2 - y_1) - k(y_2 - y_3)$$
$$m y_3'' = -k(y_3 - y_2)$$

reduction of fns

let

$$x_0 = y_0 - E\cos\omega t \equiv 0$$
$$x_1 = y_1 - E\cos\omega t$$
$$x_2 = y_2 - E\cos\omega t$$
$$x_3 = y_3 - E\cos\omega t$$

$$x_0 \equiv 0$$
$$m x_1'' = m y_1'' + mE\omega^2\cos\omega t$$
$$m x_2'' = m y_2'' + mE\omega^2\cos\omega t$$
$$m x_3'' = m y_3'' + mE\omega^2\cos\omega t$$

so

$$m x_1'' = -k(x_1 - 0) - k(x_1 - x_2) + mE\omega^2\cos\omega t$$
$$m x_2'' = -k(x_2 - x_1) - k(x_2 - x_3) + mE\omega^2\cos\omega t$$
$$m x_3'' = -k(x_3 - x_2) + mE\omega^2\cos\omega t$$

so the x's are the displacements from equilibrium as measured by an observer on the ground

Notice $y_i - y_j = x_i - x_j$

$$\boxed{\begin{bmatrix} x_1'' \\ x_2'' \\ x_3'' \end{bmatrix} = \begin{bmatrix} -2k/m & k/m & 0 \\ k/m & -2k/m & k/m \\ 0 & k/m & -k/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + E\omega^2\cos\omega t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

it's as if all floors are being forced – book calls this inertial forcing due to frame of reference