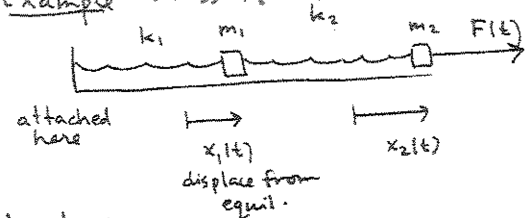
Math 2280-1
Tuesday March 1

§ 4.1, 4.3

- finish damped oscillator re-interpretation as 1^{st} order system, page 4 Monday
- spring system (and conversion below), then general discussion.

Example: mass-spring systems (§5.3)



Newton:

$$m_1 x_1''(t) = k_2 \underbrace{(x_2 - x_1)}_{\text{2nd spring is stretched this much}} - k_1 x_1$$

$$m_2 x_2''(t) = -k_2 (x_2 - x_1) + F(t)$$

e.g. $m_1 = 2$, $m_2 = 1$, $k_1 = 4$, $k_2 = 2$, $f(t) = 40 \sin 3t$ $\Rightarrow$

$$\begin{cases} 2x_1'' = 2(x_2 - x_1) - 4x_1 = -6x_1 + 2x_2 \\ x_2'' = -2(x_2 - x_1) + 40\sin 3t = 2x_1 - 2x_2 + 40\sin 3t \end{cases}$$

or
$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 40\sin 3t \end{bmatrix}$$

- What's a natural IVP for this 2^{nd} order system of 2 linear DE's?
  (We won't solve it today - see § 7.4)

note: Any system of DE's can be converted into an equivalent system of 1^{st} order DE's, by introducing extra unknown functions.

(1)
$$\boxed{\begin{aligned} x_1'' &= -3x_1 + x_2 \\ x_2'' &= 2x_1 - 2x_2 + 40\sin 3t \end{aligned}}$$

Let $y_1 = x_1'$

$y_2 = x_2'$

(2)
$$\boxed{\begin{aligned} x_1' &= y_1 \\ y_1' &= -3x_1 + x_2 \\ x_2' &= y_2 \\ y_2' &= 2x_1 - 2x_2 + 40\sin 3t \end{aligned}}$$

solutions of (1) yield solutions of (2) and vise versa. CHECK!!

This correspondence is sometimes useful.

We'll solve systems like this in §5.3

<u>Def</u> A 1^{st} order system of DE's of the form

$$* \quad \frac{d\vec{x}}{dt} - A(t)\vec{x} = \vec{f}(t) \qquad A(t) \text{ an } n\times n \text{ matrix}$$

is called a <u>linear</u> 1^{st} order system.
If $\vec{f}(t) \equiv 0$ it is called <u>homogeneous</u>

Notice (check!)

$$L(\vec{x}(t)) := \frac{d\vec{x}}{dt} - A(t)\vec{x}(t)$$ is <u>linear</u> (domain = continuously diff'ble vector valued fns $\vec{x}(t)$; codomain = cont. vector valued fns)

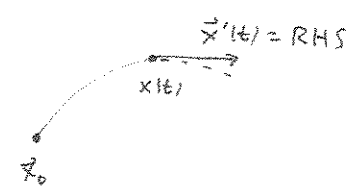
<u>Corollary</u>: The general sol'n to $*$ is

$$\vec{x}(t) = \vec{x}_H(t) + \vec{x}_p(t)$$

where $\vec{x}_p(t)$ is a particular sol'n to $*$ & $\vec{x}_H(t)$ is the general sol'n to the homogeneous DE
$$\frac{d\vec{x}}{dt} = A(t)\vec{x}.$$

<u>Theorem</u> If $A(t)$ and $\vec{f}(t)$ are defined and continuous on an interval I, and $t_0 \in I$, then the IVP

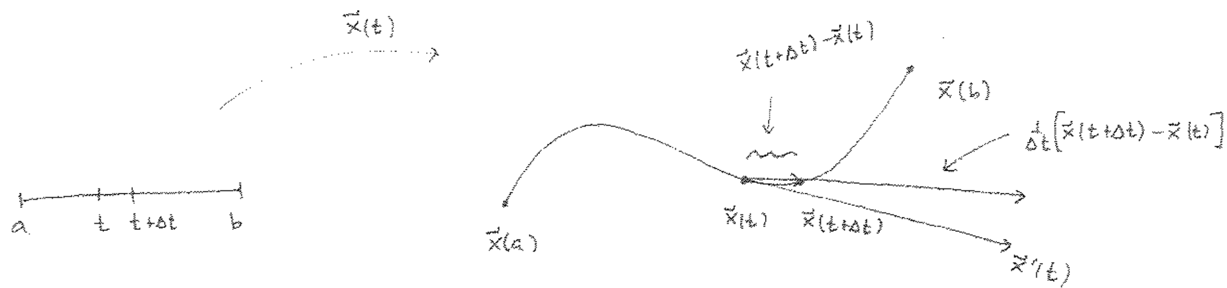
$$\begin{cases} \frac{d\vec{x}}{dt} = A(t)\vec{x} + \vec{f}(t) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$



always has a unique sol'n on I

<u>proof</u>: omitted, but the idea is you know where to start at time t_0, and you know your tangent vector $\vec{x}'(t)$ if you know where you are and what time it is, so you know where to go. The linear nature of this DE turns out to guarantee local (in time) $\exists!$, and that the solution can't blow up in finite time, so that it exists $\forall t \in I$

Digression/review (2210 or 1260), about the meaning of $\vec{x}'(t)$



$$\vec{x} : [a,b] \longrightarrow \mathbb{R}^n$$

domain codomain

$$\vec{x}'(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\vec{x}(t+\Delta t) - \vec{x}(t)\right)$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \begin{bmatrix} x_1(t+\Delta t) - x_1(t) \\ x_2(t+\Delta t) - x_2(t) \\ \vdots \\ x_n(t+\Delta t) - x_n(t) \end{bmatrix}$$

$$= \lim_{\Delta t \to 0} \begin{bmatrix} \frac{x_1(t+\Delta t) - x_1(t)}{\Delta t} \\ \vdots \\ \frac{x_n(t+\Delta t) - x_n(t)}{\Delta t} \end{bmatrix}$$

$$= \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}.$$

We called $\vec{x}'(t)$ the tangent vector or velocity vector (if $\vec{x}(t)$ was particle motion).

Thus the first order IVP solution to

$$\begin{cases} \frac{d\vec{x}}{dt} = \vec{F}(t, \vec{x}) \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

can be interpreted as ~~starting~~ the particle motion starting at $\vec{x}_0$, for which the "velocity" vector is known, in terms of current position and time, via $\vec{F}(t,\vec{x})$.

<u>Cor 1</u> The solution space to the homogeneous system

$$\frac{d\vec{x}}{dt} - A(t)\vec{x} = \vec{0}$$

is n-dim'l

Proof: Let V be the solution space to this system, which is a subspace of differentiable vector-valued functions $\vec{F}(t)$ defined for $t \in I$ (because V is closed under addition & scalar multiplication).

Let $t_0 \in I$.
Define $T: V \to \mathbb{R}^n$ by $T(\vec{x}(t)) := \vec{x}(t_0) \in \mathbb{R}^n$.
T is linear, and is an isomorphism (1-1 & onto) by the $\exists!$ theorem. Thus $\dim(V) = \dim(\mathbb{R}^n) = n$ ∎

<u>Def</u> If $\vec{y}_1(t), \ldots \vec{y}_n(t)$ are vector-valued functions with image in $\mathbb{R}^n$

then $\left[\vec{y}_1 \middle| \vec{y}_2 \middle| \cdots \vec{y}_n\right]$ is the <u>Wronskian matrix</u>
and its det, called $W(\vec{y}_1, \ldots \vec{y}_n)$ is the <u>Wronskian</u>,

<u>Theorem</u>: If $\{\vec{y}_1, \ldots \vec{y}_n\}$ each solve $\frac{d\vec{x}}{dt} - A(t)\vec{x} = \vec{0}$ on I

then either $W(y_1, \ldots y_n) \equiv 0$ on I
or $W(y_1, \ldots y_n) \neq 0 \quad \forall t \in I$.

The first case ~~is~~ means $\{y_1, \ldots y_n\}$ are dependent, the second case means they're independent.

proof: If $\det\left[\vec{y}_1 \middle| \vec{y}_2 \middle| \cdots \vec{y}_n\right] \neq 0$ at t_0 the every IVP at t_0 has a unique soltn $x(t) = c_1\vec{y}_1 + c_2\vec{y}_2 + \ldots + c_n\vec{y}_n$

($\vec{c}$ solves $\left[\vec{y}_1(t_0) \middle| \cdots \vec{y}_n(t_0)\right]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{b}$ where $\vec{b}$ is the given initial value vector $\vec{x}(t_0)$)

In this case $\{\vec{y}_1, \vec{y}_2, \ldots \vec{y}_n\}$ are a basis for the soltn space V, so for any (other) $t_1 \in I$,
$\text{span}\{\vec{y}_1(t_1), \vec{y}_2(t_1), \ldots \vec{y}_n(t_1)\} = \mathbb{R}^n$ because every IVP at t_1 has a soltn $\vec{x}(t) = c_1\vec{y}_1 + c_2\vec{y}_2 + \ldots + c_n\vec{y}_n$.

thus $\det\left[\vec{y}_1 \middle| \vec{y}_2 \middle| \cdots \vec{y}_n\right] \neq 0$ at t_1 as well. ∎

Cor 2 (the real reason $\exists!$ theorem holds for n^{th} order linear DE IVP)

Let $L(y) := y^{(n)} + a_{n-1}(t)\, y^{(n-1)} + \cdots + a_1(t)\, y' + a_0(t)\, y$, $a_j \in C(I)$

$f \in C(I)$

Let $t_0 \in I$, $\vec{b} = [b_0, b_1, \ldots b_{n-1}] \in \mathbb{R}^n$

Then $\exists!$ soltn $y(t)$ to IVP: , $y \in C^n(I)$.

$$\text{IVP1} \begin{cases} L(y) = f & \text{in } I \\ y(t_0) = b_0 \\ y'(t_0) = b_1 \\ \vdots \\ y^{(n-1)}(t_0) = b_{n-1} \end{cases}$$

proof: This IVP is equivalent to the first order system IVP for $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$:

$$\text{IVP2} \begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_n' = f - a_{n-1}x_{n-1} - a_{n-2}x_{n-2} - \cdots - a_0 x_1 \\ x_1(t_0) = b_0 \\ x_2(t_0) = b_1 \\ \vdots \\ x_n(t) = b_{n-1} \end{cases}$$

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 & \cdots & \\ 0 & 0 & 1 & \cdots & \\ & & & & \vdots \\ 0 & 0 & \cdots & - & 1 \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f \end{bmatrix}$$

i.e. $\exists!\ y$ sol'n to IVP 1

iff

$y = x_1$, $y' = x_2$, $\ldots$, $y^{(n-1)} = x_n$ is ! soltn to IVP2

Thus Cor 2, the Chapter 3 $\exists!$ theorem,
follows from the linear systems $\exists!$ theorem in Chapter 4 (page 1 theorem).