

Math 2280-1/2250-4

Wed. Jan 26

§2.3: improved velocity/acceleration models
(and §2.2: harvesting a logistic population) ← yesterday's notes

In calc: (and §1.2)

↑ y $m \frac{dv}{dt} = F_G = -mg$

$$\frac{dv}{dt} = -g$$

$$v = -gt + v_0$$

$$y = -\frac{1}{2}gt^2 + v_0t + y_0$$

add resistance?

$$m \frac{dv}{dt} = F_G + F_f$$



$$|F_f| \approx k|v|^p \quad 1 \leq p \leq 2 \text{ empiric}$$

p=1 ("linear" model)

$$m \frac{dv}{dt} = -mg - kv$$

force will cause an acceleration opposite to direction of motion

... this is low speed "linearization"

if $F_f = F(v) = F(0) + F'(0)v + \frac{1}{2!}F''(0)v^2 + \dots$

0

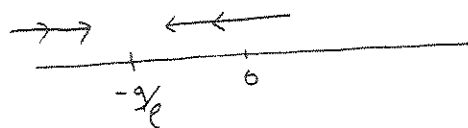
for $|v|$ small, "negligible"

$$\frac{dv}{dt} = -g - ev \quad e := \frac{k}{m}$$

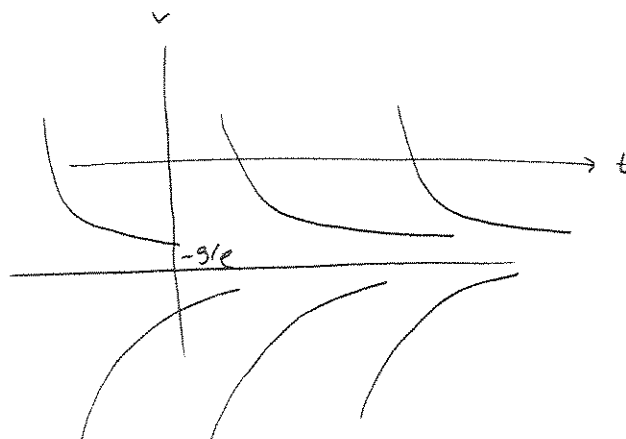
$$= -e(v + \frac{g}{e})$$

equil soltn: $v = -\frac{g}{e}$

phase portrait:



slope field:



analytic sol'n

(2)

$$\frac{dv}{dt} = -e(v + \frac{g}{e})$$

$$\frac{dv}{v + g/e} = -e dt$$

$$\ln|v + g/e| = -et + C \quad t=0 \Rightarrow C = \ln|v_0 + g/e|$$

$$= -et + \ln|v_0 + g/e|$$

$$|v + g/e| = |v_0 + g/e| e^{-et}$$

$$v + g/e = (v_0 + g/e) e^{-et} \quad (\text{why?})$$

$$\boxed{v = \underbrace{-g/e}_{v_T} + (v_0 + g/e) e^{-et}}$$

"terminal velocity"

$$y = \int v(t) dt = t v_T + \left(\frac{v_0 - v_T}{-e} \right) e^{-et} + C$$

$$\boxed{y = t v_T + \left(\frac{v_0 - v_T}{e} \right) (1 - e^{-et}) + y_0}$$

$$y_0 = \frac{v_T - v_0}{e} + C$$

$$\text{so } C = y_0 + \frac{v_0 - v_T}{e}$$

Examples 1 & 2 p. 98-100

crossbow bolt

$$v_0 = 49 \text{ m/s}$$

$$g = 9.8 \text{ m/s}^2$$

$$y_0 = 0$$

no friction

$$y = -4.9 t^2 + 49t$$

$$v = -9.8 t + 49$$

$$\text{max ht at } t = 5 \text{ sec}$$

$$y_{\text{max}} = y(5) = 49(2.5)$$

$$= 122.5 \text{ m}$$

$$\text{time aloft} = 10 \text{ sec.}$$

linear drag

$$e = .04 \quad (\text{drag coeff; empirical})$$

corresponds to

$$|v_T| = \frac{g}{e} = \frac{9.8}{.04} = 245 \text{ m/sec}$$

(you could measure this to deduce e)

$$v_0 - v_T = 49 + 245 = 294$$

so

$$v = -245 + 294 e^{-.04t}$$

$$v = 0 \text{ at } \frac{245}{294} = e^{-.04t}$$

$$t_{\text{max}} \approx 4.56 \text{ sec}$$

$$y = -245t + (294)(25)(1 - e^{-.04t})$$

$$y(t_{\text{max}}) = ?$$

When does bolt hit ground?

Computation sheet for Example 2, section 2.3

Time of maximum height:

```
> 25*ln(294.0/245);
      solve(-245+294*exp(-.04*t)=0,t); #should be same
      4.558038920
      4.558038920
```

Formulas for height and velocity functions

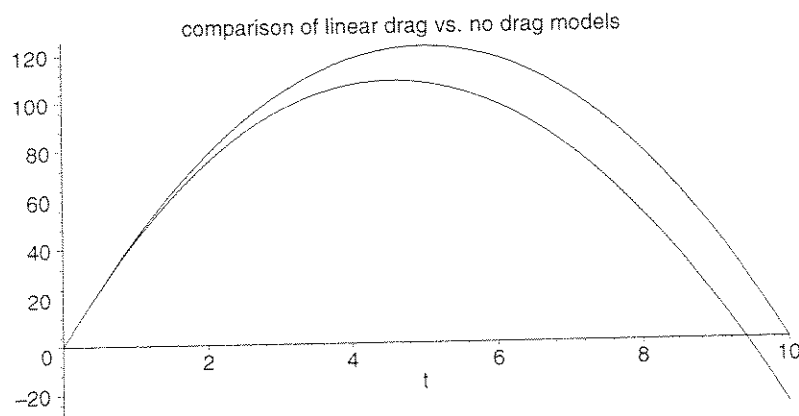
```
> v:= t -> 294*exp(-.04*t) - 245;
      v:= t -> 294 e(-0.04 t) - 245
> y:= t -> -245*t + 294*25*(1 - exp(-.04*t));
      y:= t -> -245 t + 7350 - 7350 e(-0.04 t)
```

Maximum height, return time for ground time spent falling, landing speed:

```
> y(4.558038920); #max height
      108.280465
> solve(y(t)=0,t); #find when returns to ground
      9.410949931, 0.
> 9.410949931 - 4.558038920; #time descending
      4.852911011
> v(9.410949931); #speed when it lands
      -43.2273093
```

Conclusions: bolt rises for 4.56 seconds, to a height of 108.3 meters. Then it spends 4.85 seconds descending, landing with a velocity of -43.3 meters per second.

```
> with(plots):
Warning, the name changecoords has been redefined
> z:= t -> -4.9*t^2 + 49*t; #the no resistance model
      z:= t -> -4.9 t^2 + 49 t
> plot({z(t),y(t)}, t = 0..10, color=black,
      title='comparison of linear drag vs. no drag models');
```



quadratic drag is also interesting...

going up:

$$m \frac{dv}{dt} = -mg - kv^2$$

$$\frac{dv}{dt} = -g \left(1 + \frac{k}{g} v^2\right)$$

$$\frac{dv}{1 + \frac{k}{g} v^2} = -g dt \dots$$

arctan!

going down:

$$m \frac{dv}{dt} = -mg + kv^2$$

$$\frac{dv}{dt} = -g \left(1 - \frac{k}{g} v^2\right)$$

$$\frac{dv}{1 - \frac{k}{g} v^2} = -g dt$$

parfrac! (or tanh!)