

Math 2280-1
2250-4
Monday Jan. 24

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Maple Intro's
today LCB 115
3-4 p.m.
5-6

Recall, we were discussing the logistic population model

$$\frac{dP}{dt} = kP(M-P) \quad k, M > 0. \quad M \text{ is "the carrying capacity" of the environment supporting the population}$$

e.g. $\frac{dP}{dt} = 2P(3-P)$

isoclines = ?

predicted long time behavior with positive initial populations?

analyze (formulas) solutions: separate variables! (we did most of this on Friday)

$$\begin{cases} \frac{dP}{dt} = kP(M-P) \\ P(0) = P_0 \end{cases}$$

$$\int \frac{dP}{P(M-P)} = \int k dt$$

$$\frac{1}{M} \int \left(\frac{1}{P} + \frac{1}{M-P} \right) dP = kt + C \quad (\text{Partial fractions!})$$

$$\frac{1}{M} [\ln|P| - \ln|M-P|] = kt + C$$

$$\frac{1}{M} \ln \left| \frac{P}{M-P} \right| = kt + C$$

$$\ln \left| \frac{P}{M-P} \right| = Mkt + \tilde{C}$$

$$\left| \frac{P}{M-P} \right| = e^{\tilde{C}} e^{Mkt}$$

$$\frac{P}{M-P} = C e^{Mkt} \quad (C = \frac{e^{\tilde{C}}}{1 - \tilde{C}})$$

$P(0) = P_0$, so

$$\frac{P}{M-P} = \frac{P_0}{M-P_0} e^{Mkt}$$

$$P = \left(\frac{P_0}{M-P_0} \right) e^{Mkt} (M-P)$$

$$P \left[1 + \frac{P_0}{M-P_0} e^{Mkt} \right] = \frac{MP_0}{M-P_0} e^{Mkt}$$

$$P = \frac{\left(\frac{MP_0}{M-P_0} \right) e^{Mkt}}{1 + \left(\frac{P_0}{M-P_0} \right) e^{Mkt}}$$

so,!

$$P(t) = \frac{MP_0}{(M-P_0)e^{-Mkt} + P_0}$$

long time behavior?

M is called the carrying capacity in logistic models.
Why?

• Now discuss the
U.S. population model - Maple notes
from Friday

Before discussing more population (and also velocity-acceleration) models,
let's talk about the general language that gets used...

§2.2: A general 1st order DE $\frac{dx}{dt} = f(t, x)$

is called autonomous if $\frac{dx}{dt} = f(x)$

($\frac{dx}{dt}$ only depends on x itself, not also on t)

def $x(t)$ is an equilibrium sol'n to a DE iff $x(t) \equiv C$, a constant
If the DE is autonomous and $x(t) \equiv C$ is an equilibrium sol'n, then

$$0 = \frac{dx}{dt} = f(x) = f(C) = 0.$$

And if $f(C) = 0$ then $x(t) \equiv C$ is an equilibrium sol'n.

equilibrium sol's of
 $\frac{dx}{dt} = f(x)$ are exactly the
fns $x(t) \equiv C$
where $f(C) = 0$

example

$$\frac{dx}{dt} = kx(M - x)$$

$x \equiv 0$
 $x \equiv M$ are the equil. sol'n's

example find the equil. sol'n's of

$$\frac{dx}{dt} = x^3 + 2x^2 + x$$

Def Let c be an equilibrium sol'n for a deq'n. Then

c is stable iff $\forall \varepsilon > 0 \exists \delta > 0$ s.t. for sol'ns $x(t)$ with $|x(0) - c| < \delta$
we have $|x(t) - c| < \varepsilon \quad \forall t > 0$

(sol'ns which start close enough to c stay arbitrarily close to it.)

c is unstable if it is not stable

example: Make the slope field (or alternately, sketch a sufficient sample of sol'n graphs) and phase portrait for

$$\frac{dx}{dt} = x^3 + 2x^2 + x$$

and discuss stability of the equil sol'ns $x=0, x=-1$

example find equilibria and discuss stability for $\frac{dx}{dt} = 3x - x^2$

Def c is called asymptotically stable (stronger requirement than stable)

iff $\exists \delta > 0$ s.t. $|x(t_0) - c| < \delta \Rightarrow \lim_{t \rightarrow \infty} x(t) = c.$

- Are any of the equilibria you found above asymptotically stable?

Theorem Consider the autonomous differential equation

$$\frac{dx}{dt} = f(x)$$

with f and $\frac{\partial f}{\partial x}$ continuous. (so local existence and uniqueness theorem holds).

Let $f(c)=0$ so $x(t) \equiv c$ is an equilibrium sol'n.

Suppose c is an isolated zero of f , i.e c is in an open interval I ,
 $c \in I$ and f has no other zeroes in I . Then

$$\begin{array}{c} f < 0 & f = 0 & f > 0 \\ - - - & | & + + + \\ & c & \end{array} \Rightarrow \begin{array}{c} \leftarrow & | & \rightarrow \\ & c & \end{array} \quad \text{unstable}$$

$$\begin{array}{c} f > 0 & f = 0 & f < 0 \\ + + + & | & - - - \\ & c & \end{array} \Rightarrow \begin{array}{c} \rightarrow & | & \leftarrow \\ & c & \end{array} \quad \text{asymptotically stable}$$

$$\begin{array}{c} f > 0 & f = 0 & f > 0 \\ + + + & | & + + + \\ & c & \end{array} \Rightarrow \begin{array}{c} \rightarrow & | & \rightarrow \\ & c & \end{array} \quad \text{unstable} \quad (\text{half stable})$$

$$\begin{array}{c} f < 0 & f = 0 & f < 0 \\ - - - & | & - - - \\ & c & \end{array} \Rightarrow \begin{array}{c} \leftarrow & | & \leftarrow \\ & c & \end{array} \quad \text{unstable} \quad (\text{half stable}).$$

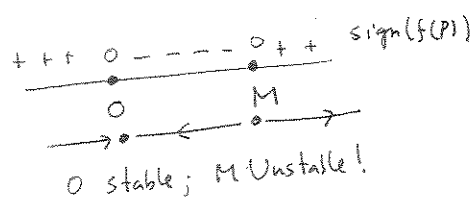
you can prove this theorem with Calculus !!
 (want to try?)

More "real" examples

doomsday extinction:

$$\frac{dP}{dt} = aP^2 - bP \quad a, b > 0.$$

$$= kP(P-M) \quad k=a, -kM=b$$



proportional to

↓

e.g. if $\beta(t)$ (fertility rate) $\sim P$

(e.g. herd roams in a fixed area and mates randomly upon meeting opposite-sex mate)

then $B(t) = aP^2$;

perhaps $D(t) = -bP$.

If $0 < P_0 < M$ then $\lim_{t \rightarrow \infty} P(t) = 0$ (so in real life, extinction)

If $P_0 > M$ then $\exists t_1 < \infty$ s.t. $\lim_{t \rightarrow t_1} P(t) = +\infty$ (doomsday!)

example

$$\begin{cases} \frac{dx}{dt} = x(x-1) \\ x(0) = 2 \end{cases}$$

Find the time of doomsday.

(ans $t = \ln 2$!)

