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Math 2280-1/2250-4

Wed Jan 19

91.5 Linear DE's cont'd.

Recall a 1st-order DE is linear if it can be written in the form

$$y' + P(x)y = Q(x)$$

$$y = y(x)$$

Solve via integrating factor

$$e^{\int P(x)dx} [y' + P(x)y] = e^{\int P(x)dx} Q(x)$$

$$[e^{\int P(x)dx} y]' = e^{\int P(x)dx} Q(x)$$

write $\bar{P}(x) = \int P(x)dx$ for an antideriv. choice

$$\bar{P}' = P$$

$$[e^{\bar{P}(x)} y]' = e^{\bar{P}(x)} Q(x)$$

antidiff

$$e^{\bar{P}(x)} y(x) = \int e^{\bar{P}(x)} Q(x) dx + C.$$

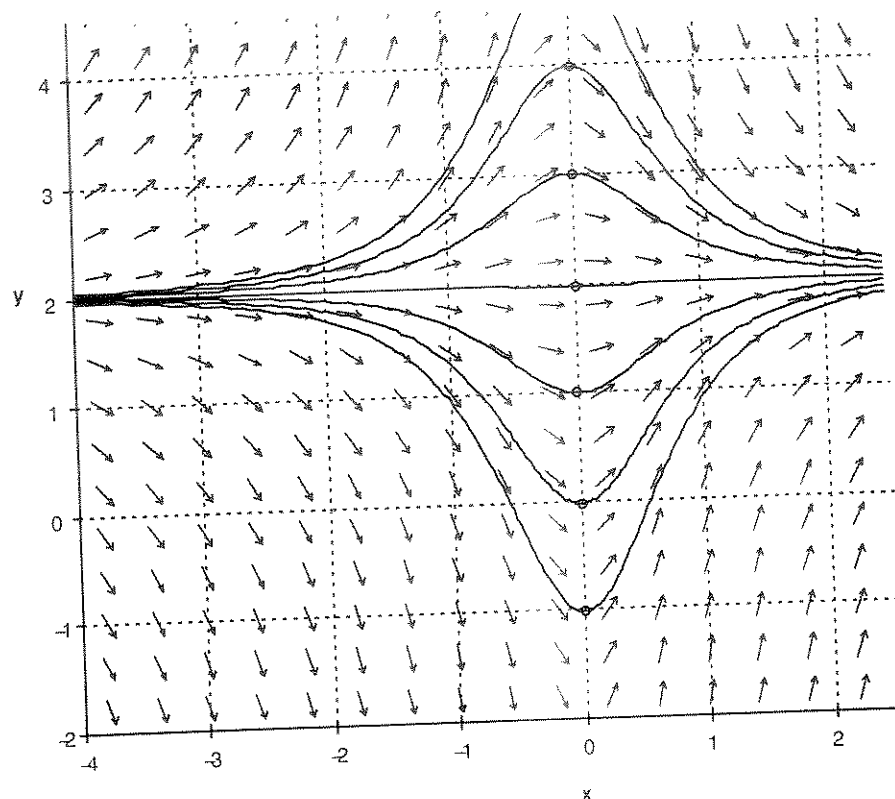
$$y(x) = e^{-\bar{P}(x)} \int e^{\bar{P}(x)} Q(x) dx + C e^{-\bar{P}(x)}$$

adjust C to solve any IVP

Exercise 1: p 50-51. Solve

$$\frac{dy}{dx} = \frac{6x - 3xy}{x^2 + 1}$$

be rewriting it as a linear DE.
Notice it is not separable



ans $y = 2 + C(x^2 + 1)^{-3/2}$

Notice, for DE

$$\text{IVP} \begin{cases} \frac{dy}{dx} + P(x)y = Q(x) \\ y(x_0) = x_0 \end{cases}$$

If $P(x), Q(x)$ are continuous on the interval I , with $x_0 \in I$,

then $\int P(x)dx$ exists (e.g. $\int_{x_0}^x P(r)dr$ is one.)

and $\int Q(x)e^{\int P(x)dx} dx$ exists, so our page 1 formula

always yields a soltn. $\exists!$ Theorem guarantees its unique on the entire interval ($\frac{\partial f}{\partial y} = -P(x)$ is cont.)

- for linear DE solution exists & is unique on entire interval, so in particular

- no vertical asymptotes

- different solution curves can't intersect.

- soltn doesn't just "stop" at a vertical point.

this is a theorem! p. 51

• Example: mixing problems (p. 51) applies also to pharmacology & environment

r_i = rate in ℓ/s
 c_i = concentration in gm/ℓ



r_o = rate out ℓ/s
 c_o = concentration out gm/ℓ

$V(t)$ = volume in tank at time t (ℓ)

$x(t)$ = amount of solute in tank (gm)

$c(t) = \frac{x(t)}{V(t)}$ $\frac{gm}{\ell}$ (average) concentration in tank

Assume mixture is uniform so $c(t)$ is spatially constant

$$\frac{dV}{dt} = r_i - r_o \quad \text{if } r_i = r_o \rightarrow V(t) = V_0 + \int_0^t (r_i(s) - r_o(s)) ds \quad \S 1.2$$

$$\frac{dx}{dt} = r_i c_i - r_o c_o \quad \frac{gm}{s}$$

by "well mixed" model

$$= r_i c_i - r_o \frac{x(t)}{V(t)}$$

$$\boxed{\frac{dx}{dt} + \frac{r_o}{V(t)} x(t) = r_i c_i}$$

$\uparrow$ $\uparrow$
 $P(t)$ $Q(t)$

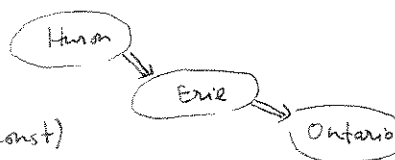
Example 4 p. 52 Lake Erie pollution

$$V(t) = 480 \text{ km}^3$$

$$r = r_i = r_o = 350 \text{ km}^3/\text{year} \quad (\text{so } V \text{ is const})$$

$c_i = c$ = pollutant conc. of lake Huron

$$x_0 = (5c)V \quad (\text{read problem})$$



When will Erie pollution concentration have decreased to twice that of Huron?

i.e. solve, and analyze

$$\begin{cases} \frac{dx}{dt} = r_i c_i - r_o c_o \\ x(0) = 5cV \end{cases}$$

$$\frac{dx}{dt} = rc - r \frac{x}{V}$$

for fun, notice this DE is also separable.
Can you solve it that way?

$$\frac{dx}{dt} + \frac{r}{V} x = rc$$

$$(e^{\frac{r}{V}t} x)' = rc e^{\frac{r}{V}t}$$

$$e^{\frac{r}{V}t} x = rc \frac{V}{r} e^{\frac{r}{V}t} + C$$

$$x = cV + C e^{-\frac{r}{V}t}$$

$$x(0) = 5cV \Rightarrow C = 4cV$$

$$x(t) = cV + 4cV e^{-\frac{r}{V}t}$$

$$\text{set } x(T) = 2cV = cV + 4cV e^{-\frac{r}{V}T}$$

$$.25 = e^{-\frac{r}{V}T}$$

$$-\frac{r}{V}T = \ln(.25)$$

$$T = -\ln(.25) \frac{V}{r}$$

$$= \ln(4) \frac{480}{350} \text{ years}$$

$$\boxed{\approx 1.90 \text{ years}}$$