Math 2250-4
2280-1

Tuesday Jan. 18 §1.4-1.5

Separable DE modeling/experiment day + intro to linear DE's
&
Introduction to MAPLE.

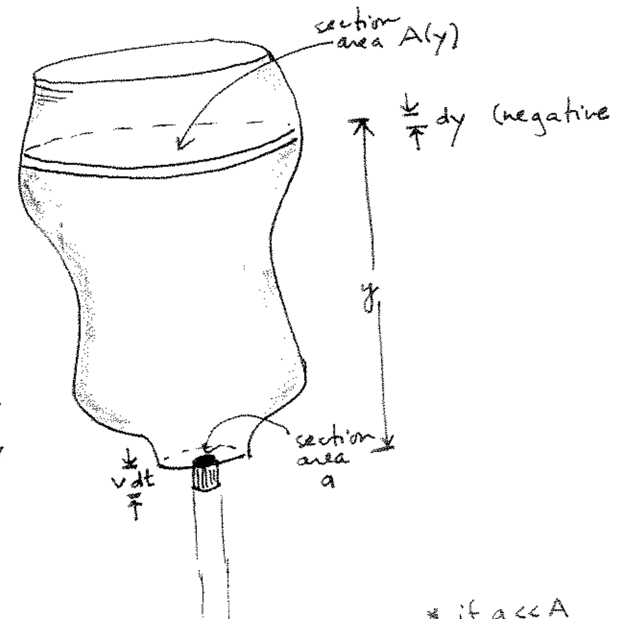
In §1.4 text & hw there are exponential growth/decay + Newton's law of cooling examples. These should be review for you, especially after last week's examples.

Here's a model that might be new to you:

Toricelli's Law for draining tanks

Let a tank be draining water. Let $y(t)$ be the water height above the drain. Then the speed v with which the water leaves the tank is given by

$$\boxed{v = \sqrt{2gy}}$$ g = acceleration of gravity.



derivation neglecting friction yields a conservative system for which

$$KE + PE = \text{const}$$

in a small time interval dt there is a (negative) change of water mass in the tank, dM.

An equal mass has drained from the drain at speed v.

By conservation of energy,

loss in PE = gain in KE

$$\boxed{\cancel{|dM|}\, g\, y = \tfrac{1}{2} \cancel{|dM|}\, v^2}\ *$$

thus $\boxed{v = \sqrt{2gy}}$

* if $a \ll A$ we may neglect the KE of the slow moving water inside the vessel.

Toricelli's Law yields a separable DE for height $y(t)$:

$$\left.\begin{aligned} dM &= \rho A(y)\,dy \qquad \rho = \text{density } (1\text{ g/cc}) \\ dM &= \rho a(-v\,dt) \qquad (\text{measured at bottom}) \end{aligned}\right\}$$

$$A(y)\,dy = -a v\,dt$$

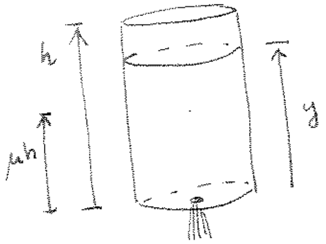
$$A(y)\,dy = -a\sqrt{2gy}\,dt \qquad \text{Toricelli}$$

$$\boxed{A(y)\frac{dy}{dt} = -k\sqrt{y}} \qquad k = a\sqrt{2g}$$

Experiment!!

Exercise 1 Cylindrical cistern: $A(y) = A$ const.

$$\frac{dy}{dt} = -k y^{1/2} \qquad \text{(different } k\text{)}.$$



Let T_μ be the amount of time it takes the cistern to empty from full height h, to height μh $(0 \le \mu \le 1)$

Show the time it takes to empty the tank is given by $\boxed{T_0 = \frac{T_\mu}{1-\mu^{1/2}}}$

use separation of variables!

Nalgene bottle experiment:

I marked off the bottle so that we can use $\mu = .5$

So if we time how long it takes to empty half the height, and call it $T_{1/2}$,

then the total time estimate will be

$$T_{0} = \frac{1}{1-\sqrt{.5}} T_{.5} \approx (3.41)\, T_{.5}$$

Experiment

$T_{1/2} =$

$(3.41)\, T_{1/2} =$

$T_0 =$

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This is a Maple 12 document. As you can see it's a mixture of text and mathematics. I've created it as a "Maple Document", as opposed to "Maple Worksheet." I will use the menu bar at the top of Maple to keep the unbracketed fields as text (or copied and pasted Maple work), and use the brackets with ">" prompts to do math.

1a) Toricelli Experiment: Here I'll just use Maple as a calculator which records computations into an easily usable file. These are numbers I get in my office today (Monday). We can see how they compare to what we do in class.

Model:

$> \dfrac{1}{1-\mathrm{sqrt}(.5)};$

$$3.414213563 \tag{1}$$

$> \mathit{Thalf} := 35.;$
$\mathit{Tpredict} := 3.414 \cdot \mathit{Thalf};$

$$\mathit{Thalf} := 35.$$
$$\mathit{Tpredict} := 119.490 \tag{2}$$

$> \mathit{Tact} := 60 + 53;$

$$\mathit{Tact} := 113 \tag{3}$$

$> \dfrac{\mathit{Tpredict}}{\mathit{Tact}};$ *#only accurate to within 6%:*

$$1.057433628 \tag{4}$$

>

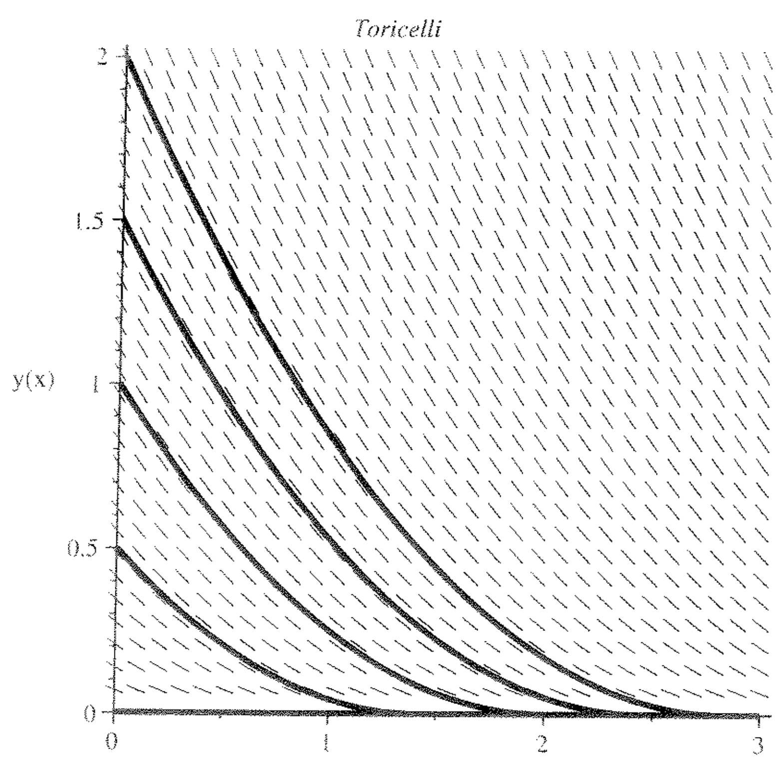
Notice that the experiment is close, but doesn't exactly match the prediction. Aside from experimental inaccuracies, can you think of places in which the model itself is inexact?

1b) Here's a Maple picture of the direction field for the differential equation. Explain where uniqueness for the IVP fails, and what it means in terms of the experiment we just did:

> *with(DETools)* : *#differential equations package*
> *deqtn* := *diff*(*y*(*x*), *x*) =-sqrt(*y*(*x*)) : *#this is the DE*
dsolve({*deqtn*, *y*(0) = 1}); *#solution!*

$$y(x) = \frac{1}{4}x^2 - x + 1 \tag{5}$$

> *DEplot*(*deqtn*, *y*(*x*), 0..3, {[*y*(0) =0], [*y*(0) = 1.],
[*y*(0) = 2.], [*y*(0) =0.5], [*y*(0) = 1.5]},
y = 0..2, *arrows* = *line*, *color* = *black*, *linecolor* = *black*,
dirgrid = [30, 30], *stepsize* = .1, *title* = 'Toricelli'); *#Maple's not as*
#cool as "dfield", but you can still draw slope fields and solution
#trajectories



.5 Linear 1st order DE's.

) $\quad \dfrac{dy}{dx} + P(x)y = Q(x)$

Notice the left side of this eqtn, $L(y) := y' + P(x)y$ is <u>linear</u>:

$L(y_1 + y_2) = L(y_1) + L(y_2)$ $\quad$ (y_1, y_2 diff'ble)

$L(cy) = cL(y)$ $\quad$ (c a const, y diff'ble)

Solution method is to multiply both sides by a non-zero fcn ("integrating factor") so that we can antidifferentiate wrt x to deduce $y(x)$:

eqtn (1) is equiv. to

.2) $\quad e^{\int P(x)dx}\left(y' + P(x)y\right) = e^{\int P(x)dx} Q(x)$ $\quad$ where $\int P(x)dx$ is any particular antideriv. of $P(x)$

equiv to

(3) $\quad \dfrac{d}{dx}\left[e^{\int P(x)dx} y\right] = \underbrace{e^{\int P(x)dx} Q(x)}$

$\Rightarrow$ (antidifferentiating)

(4) $\quad e^{\int P(x)dx} y = \int [\quad] \, dx$ $\quad$; divide by $e^{\int P(x)dx}$ to get $y(x)$

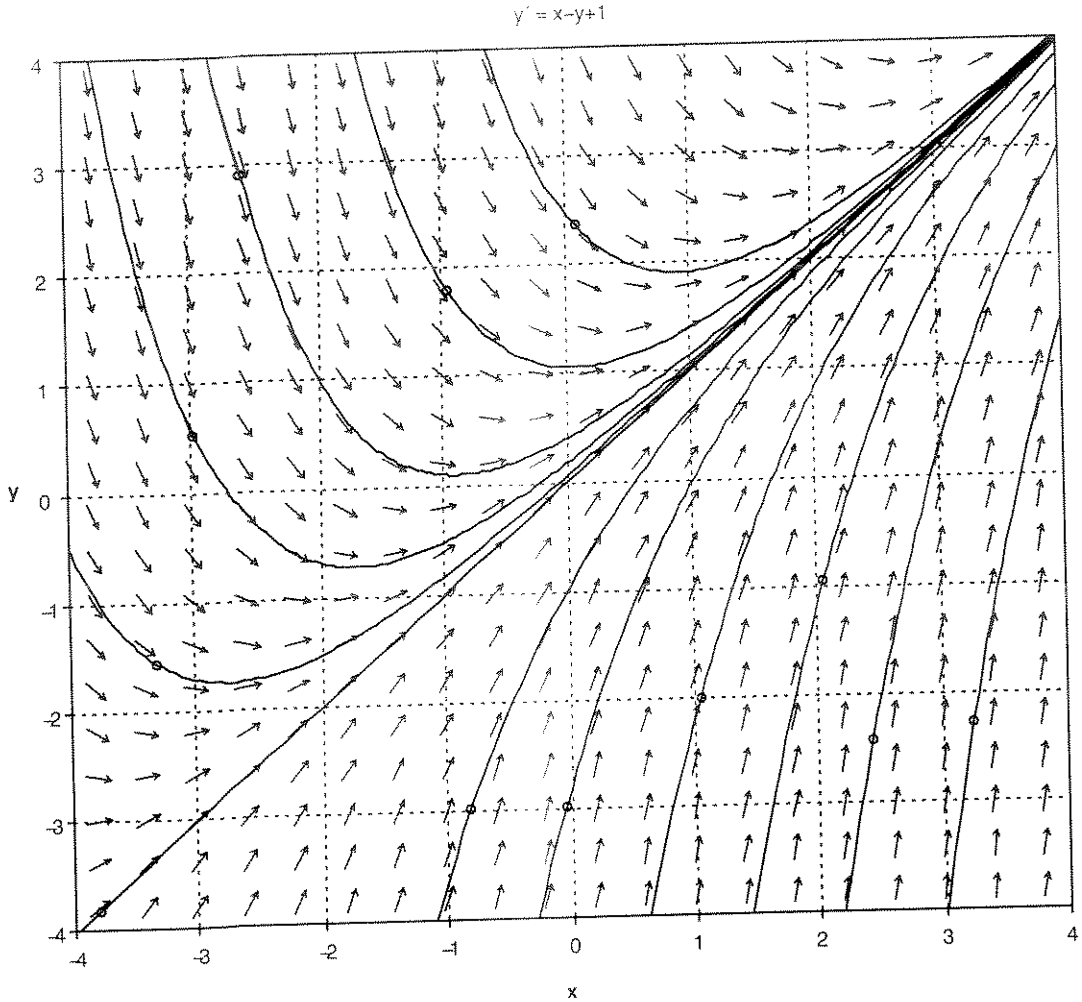
<u>Exercise 2a</u> Consider the DE

$$y' = x - y + 1$$

Rewrite it in linear form & solve as above

<u>ans</u> $y = x + Ce^{-x}$

2b) Are your answers to 2a) consistent with the slope field for
$y' = x - y + 1$?



Exercise 3 (p. 51) Solve

$$(x^2+1)\frac{dy}{dx} + 3xy = 6x$$

ans $y = 2 + C(x^2+1)^{-3/2}$