Name....Solutions....

I.D. number....

Math 2280–1
FINAL EXAM
April 29, 2011

This exam is closed–book and closed–note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. Laplace Transform and integral tables are included with this exam. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** This exam counts for 30% of your course grade. It has been written so that there are 150 points possible, and the point values for each problem are indicated in the right–hand margin. **Good Luck!**

problem	score	possible
1	______	20
2	______	20
3	______	30
4	______	20
5	______	35
6	______	15
7	______	10
total	______	150

1) Find the matrix exponentials for the following two matrices. Work one of problems using the power series definition, and the other one using the fundamental matrix solution approach (your choice). As it turns out, both methods are reasonable for both problems.

1a)

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(10 points)

$\Phi(t)\Phi(0)^{-1}$:

$|A-\lambda I| = \lambda^2 - 1 = (\lambda-1)(\lambda+1) = 0$

$\lambda = 1$

$\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}$

$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -1$

$\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}$

$\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$\Phi(t) = \begin{bmatrix} e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix}$; $\Phi(0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$e^{At} = \begin{bmatrix} e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^t+e^{-t} & e^t-e^{-t} \\ e^t-e^{-t} & e^t+e^{-t} \end{bmatrix}$

$= \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$

power series:

$A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

so $A^n = \begin{cases} A & n \text{ odd} \\ I & n \text{ even} \end{cases}$

so $e^{At} =$

$I + tA + \frac{t^2}{2}I + \frac{t^3}{3!}A + \dots$

$= \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \dots\right) I + \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots\right) A$

$= \cosh t\, I + \sinh t\, A$

$= \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$

1b)

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(10 points)

$\Phi(t)\Phi(0)^{-1}$:

$|A-\lambda I| = \lambda^2 + 1 = (\lambda+i)(\lambda-i) = 0$

$\lambda = i$

$\begin{array}{cc|c} -i & 1 & 0 \\ -1 & i & 0 \end{array}$

$R_2 = iR_1$

$\vec{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$e^{it}\vec{v} = (\cos t + i\sin t)\begin{bmatrix} 1 \\ i \end{bmatrix}$

$= \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$

$\Phi(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} = e^{At}$ because $\Phi(0) = I$

power series

$B^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$

so $B^3 = -B$

$B^4 = I$ etc.

$e^{Bt} = I + tB - \frac{t^2}{2!}I - \frac{t^3}{3!}B + \frac{t^4}{4!}I + \dots$

$= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots\right) I + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} \dots\right) B$

$= \cos t\, I + \sin t\, B$

$= \cos t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$.

2a) Use Laplace transform techniques to find the general solution to the undamped forced oscillator equation with resonance:

$$x''(t) + \omega_0^2 x(t) = F_0 \sin(\omega_0 t).$$

(10 points)

$$s^2 X(s) - s x_0 - v_0 + \omega_0^2 (X(s)) = F_0 \frac{\omega_0}{s^2+\omega_0^2}$$

$$X(s)(s^2+\omega_0^2) = F_0 \frac{\omega_0}{s^2+\omega_0^2} + s x_0 + v_0$$

$$X(s) = F_0\omega_0 \frac{1}{(s^2+\omega_0^2)^2} + \frac{s x_0}{s^2+\omega_0^2} + \frac{v_0}{s^2+\omega_0^2}$$

(table) $$x(t) = F_0\omega_0 \left(\frac{1}{2\omega_0^3}(\sin\omega_0 t - \omega_0 t\cos\omega_0 t)\right) + x_0\cos\omega_0 t + v_0/\omega_0 \sin\omega_0 t$$

$$\boxed{x(t) = \frac{F_0}{2\omega_0^2}\sin\omega_0 t - \frac{t}{2\omega_0}\cos\omega_0 t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

2b) Use Laplace transform to find the general solution to the non-resonant undamped forced oscillator equation

$$x''(t) + \omega_0^2 x(t) = F_0 \sin(\omega t)$$

$$\omega \neq \omega_0$$

(10 points)

$$s^2 X(s) - s x_0 - v_0 + \omega_0^2 X(s) = F_0 \frac{\omega}{s^2+\omega^2}$$

$$X(s)(s^2+\omega_0^2) = F_0\omega \frac{1}{s^2+\omega^2} + s x_0 + v_0$$

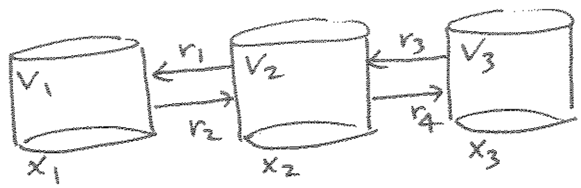
$$X(s) = F_0\omega \frac{1}{(s^2+\omega_0^2)(s^2+\omega^2)} + \frac{s x_0}{s^2+\omega_0^2} + \frac{v_0}{s^2+\omega_0^2}$$

v_0/ω_0 ω_0

$$= F_0\omega\left[\frac{1}{s^2+\omega_0^2} - \frac{1}{s^2+\omega^2}\right]\left[\frac{1}{\omega^2-\omega_0^2}\right] + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$$

$$\boxed{x(t) = \frac{F_0\omega}{\omega^2-\omega_0^2}\left[\frac{1}{\omega_0}\sin\omega_0 t - \frac{1}{\omega}\sin\omega t\right] + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

3) Consider the following three–tank configuration. Let tank i have volume $V_i(t)$ and solute amount $x_i(t)$ at time t. Well–mixed liquid flows between tanks one and two, with rates r_1, r_2, and also between tanks two and three with rates r_3, r_4, as indicated.



3a) What is the system of 6 first order differential equations governing the volumes $V_1(t)$, $V_2(t)$, $V_3(t)$ and solute amounts $x_1(t)$, $x_2(t)$, $x_3(t)$? (Hint: Although most of our recent tanks have had constant volume, we've also discussed how to figure out how fast volume is changing in input/output models.) (6 points)

$$V_1' = r_1 - r_2$$

$$V_2' = r_2 + r_3 - r_1 - r_4$$

$$V_3' = r_4 - r_3$$

$$\begin{cases} x_1' = r_1 \dfrac{x_2}{V_2} - r_2 \dfrac{x_1}{V_1} \\ x_2' = r_2 \dfrac{x_1}{V_1} + r_3 \dfrac{x_3}{V_3} - (r_1 + r_4)\dfrac{x_2}{V_2} \\ x_3' = r_4 \dfrac{x_2}{V_2} - r_3 \dfrac{x_3}{V_3} \end{cases}$$

3b) Suppose that all four rates are 100 gallons/hour, so that the volumes in each tank remain constant. Suppose that these volumes are each 100 gallons. Show that in this case, the differential equations in (2a) for the solute amounts reduce to the system

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \\ \dfrac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ x_1 - 2x_2 + x_3 \\ x_2 - x_3 \end{bmatrix}$$

(4 points)

if each $r = 100$
and each $V = 100$
then $\frac{r}{V} = 1$.
so

$$x_1' = x_2 - x_1$$
$$x_2' = x_1 + x_3 - 2x_2$$
$$x_3' = x_2 - x_3$$

which is the system above.

3c) Maple to the rescue! Maple says that

```
> with(LinearAlgebra) :
> A := Matrix(3, 3, [-1, 1, 0, 1, -2, 1, 0, 1, -1]);
  Eigenvectors(A);
```

$$A := \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \qquad \textbf{(1)}$$

Use this information to write the general solution to the system in (3b). (5 points)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

3d) Solve the initial value problem for the tank problem in (3b), assuming there are initially 10 pounds of solute in tank 1, 20 pounds in tank 2, and none in tank 3. (10 points)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 1 & 0 & -2 & 20 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$\begin{array}{l} \\ -R_1+R_2 \\ -R_1+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 0 & 1 & -3 & 10 \\ 0 & 2 & 0 & -10 \end{array}\right]$$

$$\begin{array}{l} \\ R_3/2 \\ R_2 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 0 & 1 & 0 & -5 \\ 0 & 1 & -3 & 10 \end{array}\right]$$

$$\begin{array}{l} \\ \\ -R_2+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & -3 & 15 \end{array}\right]$$

$$\begin{array}{l} \\ \\ R_3/-3 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

$$\begin{array}{l} R_1-R_3 \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 15 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

$$\begin{array}{l} R_2+R_1 \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

check: $x_1(0) = 10+5-5 = 10$
$x_2(0) = 10+10 = 20$
$x_3(0) = 10-5-5 = 0$ ✓

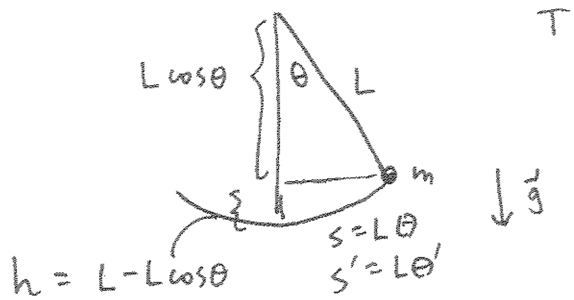
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 10 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 5e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - 5e^{-3t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

3e) What is the limiting amount of ~~salt~~ solute in each tank, as t approaches infinity? (Hint: You can deduce this answer, no matter whether you actually solved 4d, but this gives a way of partially checking your work there.) (5 points)

each tank limits to 10 lbs of solute.
from formula, the exponentially decaying terms $\to 0$ as $t \to \infty$.

also, there were initially 30 lbs of solute in this closed system, and concentrations will eventually be the same in all 3 tanks. Since the volumes of tanks are equal, then ~~tank~~ solute amt in each tank will limit to 30/3.

4) Although we usually use a mass–spring configuration to give context for studying second order differential equations, the rigid–rod pendulum also effectively exhibits several key ideas from this course. Recall that in the undamped version of this configuration, we let the pendulum rod length be L, assume the rod is massless, and that there is a mass m attached at the end on which the vertical graviational force acts with force $m \cdot g$. This mass will swing in a circular arc of signed arclength $s = L \cdot \theta$ from the vertical, where θ is the angle in radians from vertical. The configuration is indicated below.



$TE = KE + PE = \text{const}.$

$= \frac{1}{2} m \left(L\theta'(t)\right)^2 + mgL(1-\cos\theta) = \text{const}.$

thus $\frac{d}{dt} TE \equiv 0$. $TE = mL\left(\frac{1}{2}L(\theta'(t))^2 + g(1-\cos\theta)\right)$

so $0 = mL\left[L\theta'\theta'' + g\sin\theta\,\theta'\right]$

i.e. $\theta'\left[L\theta'' + g\sin\theta\right] = 0$

4a) Use the fact that the undamped system is conservative, to derive the differential equation for $\theta(t)$,

4a $\qquad \theta''(t) + \frac{g}{L} \cdot \sin(\theta(t)) = 0.$

so on intervals for which $\theta' \neq 0$, $L\theta'' + g\sin\theta = 0$

(10 points)

by continuity

$L\theta'' + g\sin\theta = 0$

4b) Explain precisely how the second order differential equation in (5a) is related to the first order system of differential equations

4b $\qquad \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} y \\ -\dfrac{g\sin(x)}{L} \end{bmatrix}$

(5 points)

If $\theta(t)$ solves 4a

let $x(t) = \theta$

$y(t) = \theta'$

then $x' = y$

$y' = \theta'' = -\frac{g}{L}\sin\theta = -\frac{g}{L}\sin x$

so $\begin{bmatrix} x \\ y \end{bmatrix}$ solves 4b

If $\begin{bmatrix} x \\ y \end{bmatrix}$ solves 4b,

let $\theta = x$.

then $\theta' = x' = y$

and $\theta'' = y' = -\frac{g}{L}\sin x = -\frac{g}{L}\sin\theta$

so θ solves 4a

4c) Find all equilibrium solutions of the non–linear system (4b). Explain what configurations of the pendulum these solutions correspond to.

(5 points)

$\theta' = y = 0$

$\sin x = 0$

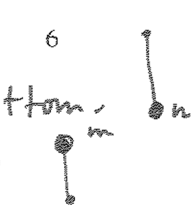
$\theta = x = n\pi,\ n \in \mathbb{Z}.$

so equil. soltns of 4b are

$\{(n\pi, 0)\}_{n\in\mathbb{Z}}$. $(\theta' = 0)$

if n is even pendulum is at rest at bottom.

if n is odd pendulum is at rest at top



5) Although we could continue with the general case in the preceding problem, let's assume that $\frac{g}{L} = 1$, so that the system in (4b) becomes

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} y \\ -\sin(x) \end{bmatrix}.$$

(By the way, this isn't such a big assumption, since one could always rescale various units so that the numerical value of $\frac{g}{L}$ in the new units is in fact equal to one.)

5a) Linearize the system above near the equilibrium solutions (which you already found in problem (4)). There are basically two different cases. What conclusions you can draw about stability at these equilibrium solutions for the non-linear system, based *only* on the analysis of the linearized problems? (12 points)

$$J = \begin{bmatrix} 0 & 1 \\ -\cos x & 0 \end{bmatrix}$$

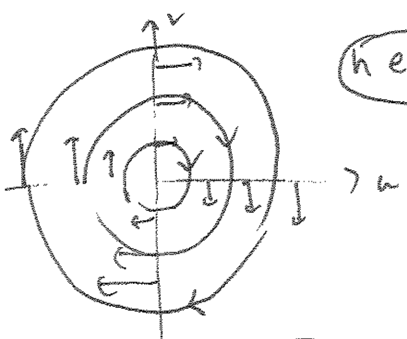
if $x = n\pi$ n even

then $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\lambda = \pm i$

stable center for linear problem with circular trajectories. See problem 1b.

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$



n even

5d

this is borderline for non-linear problem, might or might not be stable.

if $x = n\pi$ n odd

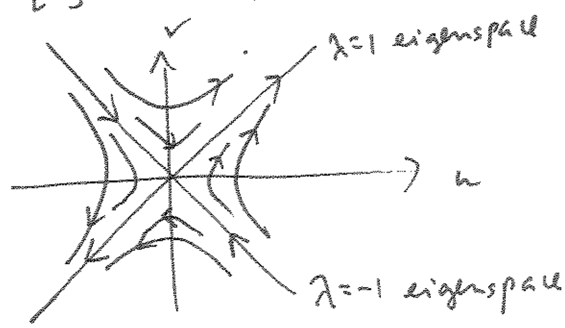
then $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\lambda = \pm 1$ saddle. unstable also for nonlinear problem.

local picture from (1b)

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = c_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



n odd

5b) To more fully understand stability for the non–linear problem, use the method related to separable differential equations, which we've used in situations like this, to prove that the solution trajectories to the nonlinear system in (5) follow the level curves of a certain function. How is this function related to your work in problem (4a)? (8 points)

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin x}{y}$$

$$y\,dy = -\sin x\,dx$$

$$\frac{1}{2}y^2 = \cos x + C$$

$$\boxed{\frac{1}{2}y^2 - \cos x = C}$$

the ~~[illegible]~~ TE function in 4a) ~~for these parameters [illegible]~~ was

$$\frac{1}{2}mL^2y^2 + mgL - mgL\cos x$$

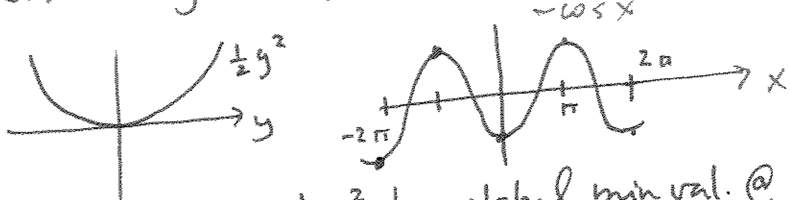
$$= mL^2\left(\frac{1}{2}y^2 - \frac{g}{L}\cos x\right) + mgL$$

5c) Using the function you found in (5b) or ideas from (4a), explain the complete equilibrium stability story for the non–linear system in (5). (5 points)

$$\underbrace{\boxed{\frac{1}{2}y^2 - \cos x}}_{}\quad \text{when } \frac{g}{L} = 1$$

so TE is const iff the sep. vars fn is const, and the two functions differ by an additive constant and a scale fact

let $f(x,y) = \frac{1}{2}y^2 - \cos x$



since $\frac{1}{2}y^2$ has global min val. @ $y=0$

& $-\frac{1}{2}\cos x$ " " " " @ $x = n\pi$ n even

deduce $f(x,y)$ has global minima at $(n\pi, 0)$ n even

and we see graphically (or with an analysis of the Hessian) that the level curves of f near these equilibria are close to elliptical loops, i.e. they close up. If (x_0, y_0) is near such a point, ~~f is at~~ $f(x_0, y_0)$ is close to -1, $f(x(t), y(t))$ stays constant, i.e. still near to -1, and so $(x(t), y(t))$ stays close to its nearby equil. sol.

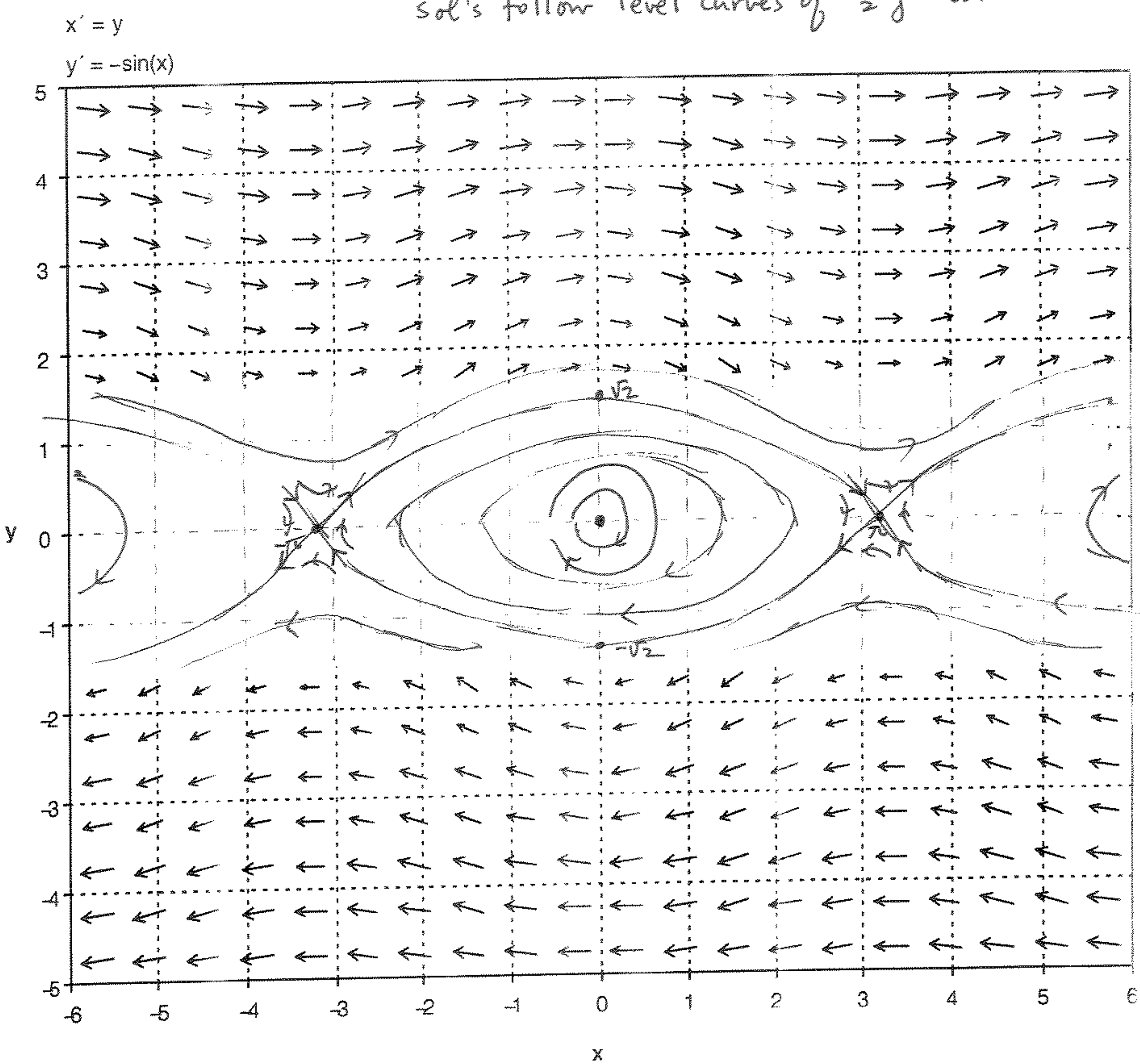
thus $(n\pi, 0)$, n even, are $\boxed{\text{stable equilibria.}}$

5d) Carefully fill in the missing parts of the phase portrait below. The best way to do this is to fill in enough solution trajectories so that their geometric shape is clear. You may use previous work from anywhere else on this exam. Make sure to explain your work. Utilize the eigendata from the linearized problems near the equilibria as appropriate.

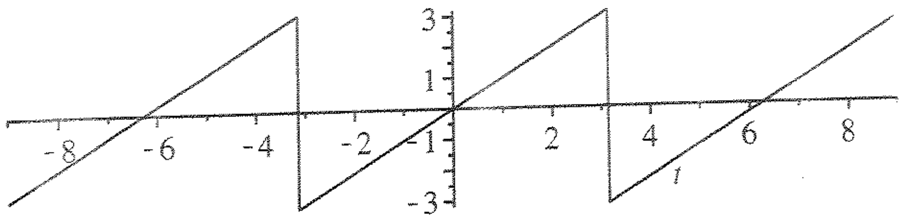
(10 points)

see linearization pictures located @ 5a

sol's follow level curves of $\frac{1}{2}y^2 - \cos x$



6a) We consider a 2 π–periodic saw–tooth function, given on the interval $(-\pi, \pi)$ by $f(t) = t$, and equal to zero at every integer multiple of π.
Here's a graph of a piece of this function:



Derive the Fourier series for $f(t)$,

$$f(t) = 2\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(n\,t)}{n}\right).$$

(10 points)

f is odd so its Fourier Series is a sine series

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(t)\sin nt}_{\text{even fn}}\,dt = \frac{2}{\pi}\int_0^{\pi} \underset{u}{t}\, \underbrace{\sin nt\,dt}_{dv}$$

(you could use the integral table).

$du = dt \quad v = -\frac{\cos nt}{n}$

$$= \frac{2}{\pi}\left[t\left(-\frac{\cos nt}{n}\right)\Big]_0^{\pi} - \underbrace{\int_0^{\pi} -\frac{\cos nt}{n}\,dt}\right]$$

antiderivative is a multiple of $\sin nt$, evaluates to zero at both endpoints.

$$b_n = \frac{2}{\cancel{\pi}}\left[\frac{1}{n}\cancel{\pi}\left(-\cos n\pi + 0\right)\right]$$

$-\cos n\pi = -(-1)^n$

$$= \frac{2}{n}(-1)^{n+1}$$

$$\text{So } f \sim 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nt}{n}$$

6b) Use the Fourier series above to explain the identity

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

(5 points)

@ $t = \frac{\pi}{2}$ series converges so

$$\frac{\pi}{2} = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin\frac{n\pi}{2}}{n}$$

$$\sin\frac{n\pi}{2} = \begin{cases} 0 & n \text{ ~~odd~~ even} \\ (-1)^{\frac{n-1}{2}} & n \text{ odd} \end{cases}$$

(alternates by odd #).

$$= 2\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right]$$

$$\text{So } (-1)^{n+1}\sin\frac{n\pi}{2} = \begin{cases} 0 & \text{here} \\ (-1)^{\frac{n-1}{2}} & n \text{ odd} \end{cases}$$

$\div 2$: $\quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

7) Consider the saw–tooth function $f(t)$ from problem 6, and the forced oscillation problem

$$x''(t) + 9\cdot x(t) = f(t). \qquad w_0 = 3$$

7a) Discuss whether or not resonance occurs. (3 points)

$$f(t) \sim 2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nt}{n}$$

since $w_0 = 3$ the $\sin 3t$ term in f will cause resonance.

7b) Find a particular solution for this forced oscillation problem. Hint: your work in problem (2) could be useful. (7 points)

let $x(t) = \sum_{n=1}^{\infty} B_n \sin nt$

$$x''(t) = \sum_{n=1}^{\infty} -n^2 B_n \sin nt$$

$$x'' + 9x = \sum_{n=1}^{\infty} B_n(-n^2+9)\sin nt = 2\sum (-1)^{n+1} \frac{\sin nt}{n}.$$

for $n \neq 3$ equating coeffs yields $B_n = (-1)^{n+1} \frac{2}{n(-n^2+9)}$

for $n = 3$ no such B_n exists because the $\sin 3t$ term is causing resonance.

from page 3 #2a we see that for $x'' + 9x = \sin 3t$, an x_p is $-\frac{t}{6}\cos 3t$

for $n=3$ we actually want to solve $x'' + 9x = \frac{2}{3}\sin 3t$

so $x_p = \frac{2}{3}\left(-\frac{t}{6}\cos 3t\right)$

$= -\frac{t}{9}\cos 3t$

thus, for our problem, an

$$x_p = \underbrace{\sum_{\substack{n=1 \\ n\neq 3}}^{\infty} (-1)^{n+1} \frac{2}{n(-n^2+9)} \sin nt}_{\text{bounded by ratio comp. test since } \sum \frac{1}{n^3} < \infty} \underbrace{- \frac{t}{9}\cos 3t}_{\text{resonating.}}$$