Name...

I.D. number...

Math 2280–1
FINAL EXAM
April 29, 2011

This exam is closed–book and closed–note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. Laplace Transform and integral tables are included with this exam. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** This exam counts for 30% of your course grade. It has been written so that there are 150 points possible, and the point values for each problem are indicated in the right–hand margin. **Good Luck!**

problem	score	possible
1	_____	20
2	_____	20
3	_____	30
4	_____	20
5	_____	35
6	_____	15
7	_____	10
total	_____	150

1) Find the matrix exponentials for the following two matrices. Work one of problems using the power series definition, and the other one using the fundamental matrix solution approach (your choice). As it turns out, both methods are reasonable for both problems.

1a)

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(10 points)

1b)

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(10 points)

2a) Use Laplace transform techniques to find the general solution to the undamped forced oscillator equation with resonance:

$$x''(t) + \omega_0^2 x(t) = F_0 \sin(\omega_0 t).$$

<div align="right">(10 points)</div>

2b) Use Laplace transform to find the general solution to the non−resonant undamped forced oscillator equation

$$x''(t) + \omega_0^2 x(t) = F_0 \sin(\omega t)$$

$$\omega \neq \omega_0$$

<div align="right">(10 points)</div>

3) Consider the following three–tank configuration. Let tank i have volume $V_i(t)$ and solute amount $x_i(t)$ at time t. Well–mixed liquid flows between tanks one and two, with rates r_1, r_2, and also between tanks two and three, with rates r_3, r_4, as indicated.

3a) What is the system of 6 first order differential equations governing the volumes $V_1(t)$, $V_2(t)$, $V_3(t)$ and solute amounts $x_1(t)$, $x_2(t)$, $x_3(t)$? (Hint: Although most of our recent tanks have had constant volume, we've also discussed how to figure out how fast volume is changing in input/output models.

(6 points)

3b) Suppose that all four rates are 100 gallons/hour, so that the volumes in each tank remain constant. Suppose that these volumes are each 100 gallons. Show that in this case, the differential equations in (2a) for the solute amounts reduce to the system

$$
\begin{bmatrix} \dfrac{dx_1}{dt} \\[2mm] \dfrac{dx_2}{dt} \\[2mm] \dfrac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
$$

(4 points)

3c) Maple to the rescue! Maple says that

$$A := \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \qquad \textbf{(1)}$$

Use this information to write the general solution to the system in (3b).

(5 points)

3d) Solve the initial value problem for the tank problem in (3b), assuming there are initially 10 pounds of solute in tank 1, 20 pounds in tank 2, and none in tank 3.

(10 points)

3e) What is the limiting amount of salt in each tank, as t approaches infinity? (Hint: You can deduce this answer, no matter whether you actually solved 4d, but this gives a way of partially checking your work there.)

(5 points)

4) Although we usually use a mass–spring configuration to give context for studying second order differential equations, the rigid–rod pendulum also effectively exhibits several key ideas from this course. Recall that in the undamped version of this configuration, we let the pendulum rod length be L, assume the rod is massless, and that there is a mass m attached at the end on which the vertical graviational force acts with force $m \cdot g$. This mass will swing in a circular arc of signed arclength $s = L \cdot \theta$ from the vertical, where θ is the angle in radians from vertical. The configuration is indicated below.

4a) Use the fact that the undamped system is conservative, to derive the differential equation for $\theta(t)$,

$$\theta''(t) + \frac{g}{L} \cdot \sin(\theta(t)) = 0.$$

(10 points)

4b) Explain precisely how the second order differential equation in (5a) is related to the first order system of differential equations

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} y \\ -\dfrac{g \sin(x)}{L} \end{bmatrix}$$

(5 points)

4c) Find all equilibrium solutions of the non–linear system (4b). Explain what configurations of the pendulum these solutions correspond to.

(5 points)

5) Although we could continue with the general case in the preceding problem, let's assume that $\frac{g}{L} = 1$, so that the system in (4b) becomes

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} y \\ -\sin(x) \end{bmatrix}.$$

(By the way, this isn't such a big assumption, since one could always rescale various units so that the numerical value of $\frac{g}{L}$ in the new units is in fact equal to one.)

5a) Linearize the system above near the equilibrium solutions (which you already found in problem (4)). There are basically two different cases. What conclusions you can draw about stability at these equilibrium solutions for the non–linear system, based *only* on the analysis of the linearized problems?

(12 points)

5b) To more fully understand stability for the non−linear problem, use the method related to separable differential equations, which we've used in situations like this, to prove that the solution trajectories to the nonlinear system in (5) follow the level curves of a certain function. How is this function related to your work in problem (4a)?

(8 points)

5c) Using the function you found in (5b) or ideas from (4a), explain the complete equilibrium stability story for the non−linear system in (5).
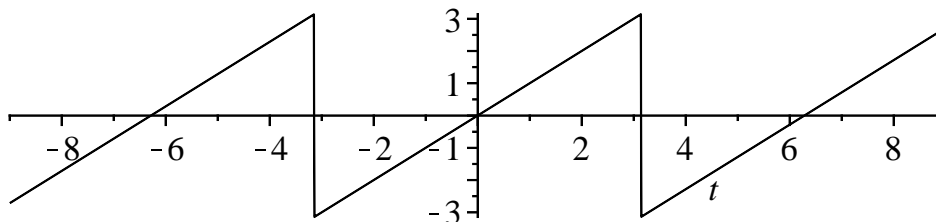
(5 points)

5d) Carefully fill in the missing parts of the phase portrait below. The best way to do this is to fill in enough solution trajectories so that their geometric shape is clear. You may use previous work from anywhere else on this exam. Make sure to explain your work. Utilize the eigendata from the linearized problems near the equilibria as appropriate.

(10 points)

6a) We consider a 2π–periodic saw–tooth function, given on the interval $(-\pi, \pi)$ by $f(t) = t$, and equal to zero at every integer multiple of π.
Here's a graph of a piece of this function:

Derive the Fourier series for $f(t)$,

$$f(t) = 2\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(n t)}{n}\right).$$

(10 points)

6b) Use the Fourier series above to explain the identity

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

(5 points)

10

7) Consider the saw–tooth function $f(t)$ from problem 6, and the forced oscillation problem

$$x''(t) + 9 \cdot x(t) = f(t).$$

7a) Discuss whether or not resonance occurs.

(3 points)

7b) Find a particular solution for this forced oscillation problem. Hint: your work in problem (2) could be useful.

(7 points)