Math 2280-1
Wed. Feb. 9

Exam 1 Next Friday 2/18
covers thru §3.5
HW for Wed. 2/16
§3.4 4, 5, 6 13, 15, 19, 23
§3.5 3 4, 17, 19 36, 37, 43, 49 50, 51, 64

We're considering constant coefficient, linear, homogeneous DE's

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

Are discussing algorithms for finding bases for the n-dim'l solution space, in terms of the characteristic polynomial

$$p(r) = r^n + a_{n-1}r^{n-1} + \dots + a_1 r + a_0$$

and its roots.

We need to finish the discussion of what to do about complex roots,
- Tuesday notes

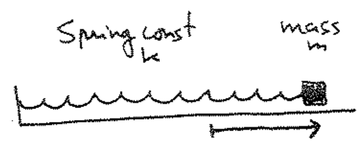
Then, begin §3.4 : applications to mechanical oscillation problems

$$\underset{\text{theory}}{y(x)} \leftrightsquigarrow \underset{\text{applications}}{x(t)}$$

Recall the linearized spring equation from earlier in the course:

$$m x'' + c x' + k x = f(t)$$

external driving force $\equiv 0$ in §3.4



$x(t)$ = displacement from equil.

$m x''$ = net forces
$\approx -kx - \mu x' + f(t)$ linear approx

We study this extremely important DE in stages:

<u>Case 1</u> <u>Free undamped motion</u>

Free: no driving force in §3.4 ; undamped: $c = 0$

$$m x'' + k x = 0$$

$$x'' + \frac{k}{m} x = 0$$

$$x'' + \omega_0^2 x = 0 \qquad \left(\omega_0 := \sqrt{\frac{k}{m}}\right)$$

if $x = e^{rt}$, $p(r) = r^2 + \omega_0^2 = 0$

$$r = \pm i\omega_0$$

$$e^{i\omega_0 t} = \cos\omega_0 t + i \sin\omega_0 t$$

$$\boxed{\begin{array}{l} x(t) = A\cos\omega_0 t + B\sin\omega_0 t \\ \quad \text{simple harmonic motion} \end{array}}$$

$$= C\cos(\omega_0 t - \alpha)$$
$$= C\cos(\omega_0(t - \delta))$$

!!

Example 1, page 189

$m = \frac{1}{2}$ kg

when $x = 2$m., force of spring is 100 N.

i.e. $2k = 100$ $k = 50$ N/m

$$\frac{1}{2}x'' + 50x = 0$$

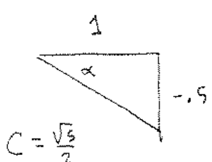
$$x'' + 100x = 0 \qquad \omega_0 = 10$$

IVP $\begin{cases} x_0 = 1 \text{ m} \\ v_0 = -5 \text{ m/s.} \end{cases}$

$$x(t) = A\cos 10t + B\sin 10t$$

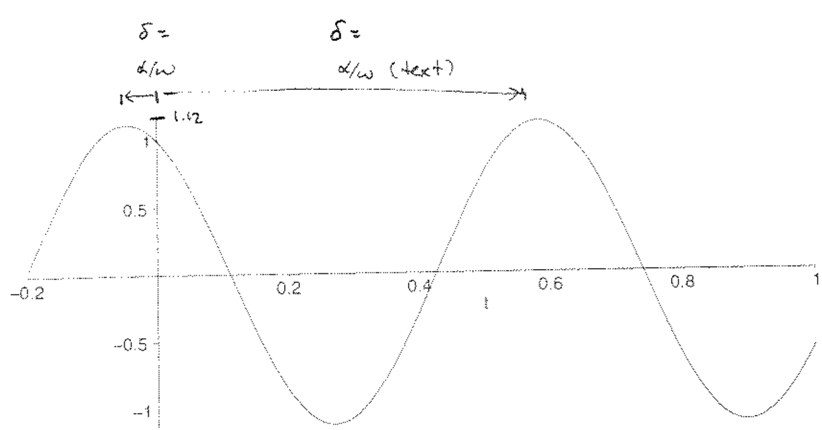
$$\boxed{x(t) = \cos 10t - .5\sin 10t}$$

$$x(t) = \frac{\sqrt{5}}{2}\cos(10t - \alpha) \approx (1.12)\cos(10t + .46)$$



$\alpha = \tan^{-1}(-.5)$
$\approx -.46$ rad.

(compare with text page 190.
says $\alpha = 5.8195$.
both answers are correct!)



Case 2 : Damping

$$m x'' + c x' + k x = 0$$

$$x'' + 2p x' + \omega_0^2 x = 0 \qquad \omega_0 = \sqrt{\tfrac{k}{m}} \text{ still}; \quad \tfrac{c}{m} = 2p; \quad p = \tfrac{c}{2m}$$

$$p(r) = r^2 + 2pr + \omega_0^2 = 0$$

$$r = \frac{-2p \pm \sqrt{4p^2 - 4\omega_0^2}}{2}$$

$$\boxed{r = -p \pm \sqrt{p^2 - \omega_0^2}}$$

Overdamped : $p^2 - \omega_0^2 > 0 \quad (c^2 > 4km)$

$$r = r_1 < r_2 < 0$$

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = e^{r_1 t}\left(c_1 + c_2 e^{(r_2 - r_1)t}\right)$$

Figure 3.4.7 p. 191

→ sol's decay exponentially to zero, cross t-axis at most 1 time.

Critically damped $p^2 - \omega_0^2 = 0 \quad (c^2 = 4km)$

$r = -p$ double root

$$x(t) = c_1 e^{rt} + c_2 t e^{rt} = e^{rt}(c_1 + c_2 t)$$

Figure 3.4.8 p. 191 → sol's decay exponentially to zero, cross t-axis at most once

Underdamped $p^2 - \omega_0^2 < 0$

set $\omega_1 = \sqrt{\omega_0^2 - p^2} < \omega_0$

$$r = -p \pm i\omega_1$$

$$e^{rt} = e^{(-p \pm i\omega)t}$$

$$x(t) = e^{-pt}(A\cos\omega_1 t + B\sin\omega_1 t)$$

$$= e^{-pt}(C\cos(\omega_1 t - \alpha))$$

Figure 3.4.9. p. 192

pseudo period $T = \frac{2\pi}{\omega_1}$

pseudo freq $f = \frac{\omega_1}{2\pi}$

pseudo ang. freq ω_1

Ce^{-pt} : " time varying amplitude

1. solution oscillates, but oscillations are damped exponentially
2. motion is slowed relative to no damping ($\omega_1 < \omega_0$).

Examples & experiments next week!

Example 2, page 192

add a little damping ($c=1$) to example 1.

$$\tfrac{1}{2}x'' + x' + 50x = 0$$

$$\begin{cases} x'' + 2x' + 100x = 0 \\ x(0) = 1 \\ x'(0) = -5 \end{cases}$$

$r^2 + 2r + 100 = 0$ or $r = \frac{-2 \pm \sqrt{4-400}}{2}$

$(r+1)^2 + 99 = 0$

$r = -1 \pm i\sqrt{99}$ $= -1 \pm i\sqrt{99}$

$$x(t) = e^{-t}\left(A\cos\sqrt{99}\,t + B\sin\sqrt{99}\,t\right)$$

← note, damping retards angular freq. $\omega_1 = \sqrt{99} \approx 9.95$

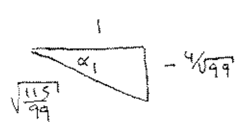
$x(0) = 1 = A$

$x'(0) = -5 = -A + \sqrt{99}\,B$

$B = -\frac{4}{\sqrt{99}}$

$$x(t) = e^{-t}\left[\cos(\sqrt{99}\,t) - \frac{4}{\sqrt{99}}\sin(\sqrt{99}\,t)\right]$$

$$= e^{-t}\left[C_1\cos(\sqrt{99}\,t - \alpha_1)\right] \approx \boxed{1.078\,e^{-t}\cos(9.95t - -.38)}$$



α_1 larger

$\alpha_1 = \tan^{-1}\left(-4/\sqrt{99}\right)$