Math 2280-1
Tuesday Feb. 8

- Carefully go through the rest of Monday's notes, which explain how to find a basis for ker L when $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0$ (a_j constant) (a_j real)

  and the characteristic polynomial

  $$p(r) := r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0$$

  has real roots only (either n distinct roots, or repeated roots.)

- Then figure out what to do when the characteristic poly has complex roots:

Theorem 8 If r is a complex root of $p(r)$, $r = a + bi$ (a, b real) $i = \sqrt{-1}$

then $y_1 = e^{ax}\cos bx$
$y_2 = e^{ax}\sin bx$
are linearly independent soltns to $L(y) = 0$.

proof: you could check this directly, but there's a more natural (and motivated) proof using complex exponentials.

Step 1 Euler's formula: $\boxed{e^{i\theta} := \cos\theta + i\sin\theta}$ (θ real)

motivation: check Taylor series!

not-needed here but good to know Step 1′: $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$ is true! (cos, sin addition angle!)

Step 2 $e^{\alpha + i\beta} := e^{\alpha}e^{i\beta} = e^{\alpha}(\cos\beta + i\sin\beta)$ (α, β real)

so $e^{(a+ib)x} = e^{ax+ibx} = e^{ax}(\cos bx + i\sin bx)$ (x real, a, b real)

Step 3 $D_x(f(x) + ig(x)) := f'(x) + ig'(x)$ f, g real fcns this is a definition

Step 4 $\quad D_x\left(e^{(a+ib)x}\right) = (a+ib)\,e^{(a+ib)x}$

this is a computation, using the definition in step 3.

i.e. if r is a complex number $D_x\left(e^{rx}\right) = re^{rx}$.

This needs to be checked!!

proof of Theorem 8 $\quad$ Let $r = a+bi$ a root of the characteristic polynomial p

Let $y = e^{rx} = e^{ax}(\cos bx) + i\,e^{ax}\sin bx$

$= y_1 + i\,y_2$

by step 4,
$L(y) = L(e^{rx}) = p(r)e^{rx} = 0. = 0 + 0i$

by Step 3, $L(y) = L(y_1 + iy_2) = L(y_1) + i\,L(y_2)$
(and linearity)

so $\underset{\displaystyle 0+0i}{\overset{\shortparallel}{0}} = \underbrace{L(y_1)}_{\substack{\text{real function,}\\ \text{since } a_i\text{'s are real}}} + i\,\underbrace{L(y_2)}_{\substack{\text{imag. fcn}\\ \text{since } a_i\text{'s are real.}}}$

Thus $L(y_1) = 0$ (equate real parts)
$L(y_2) = 0$ (equate imag. parts).

last step: Show $\begin{array}{l} y_1 = e^{ax}\cos bx \\ y_2 = e^{ax}\sin bx \end{array}$ are linearly independent:

Exercise $\quad$ Find the general solution of

$$y'' + 4y' + 5y = 0.$$

Using the ideas we've been discussing, you get the following algorithm to find all sol'ns to $L(y)=0$ for the const. coeff. linear DE of order n:

Theorem 9: Let $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y$ $\quad a_j$ const.

$p(r) = r^n + a_{n-1}r^{n-1} + \dots + a_1 r + a_0$

if r_j is a real root of p, $p(r_j)=0 \implies y = e^{r_j x}$ solves $L(y)=0$

if $(r-r_j)^{k_j}$ is a factor of $p \implies$ $y_1 = e^{r_j x}$, $y_2 = x e^{r_j x}$, $\vdots$, $y_{k_j} = x^{k_j - 1} e^{r_j x}$ are k_j lin ind sol'tns to $L(y)=0$

if $r_j = a+bi$ is a complex root, $p(a+bi)=0 \implies y_1 = e^{ax}\cos bx$, $y_2 = e^{ax}\sin bx$ are l.i. sol'tns to $L(y)=0$

if $(r-r_j)^{k_j}$ is a factor of p

also get $x e^{ax}\cos bx$, $x e^{ax}\sin bx$

$\vdots \qquad \vdots$

$x^{k_j-1} e^{ax}\cos bx$, $x^{k_j-1} e^{ax}\sin bx$

(gives $2k_j$ sol'tns for the roots r_j & $\bar{r}_j$)

In this way we can always find a basis for $\ker(L)$, (assuming we can figure out how to factor p!)

Exercise Find the general solution to

$$y''' - 5y'' + 24y' - 20y = 0 \,!$$

Applications on Wednesday, § 3.4

should be fun

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