

Math 2280-1
Monday Feb. 7

- Exercise 3, Friday's notes. Discuss how to solve IVP for this non-homogeneous problem.

Constant coefficient n^{th} -order linear operators L

$$\text{i.e. } L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_2y'' + a_1y' + a_0y \quad a_j = \text{const} \quad 0 \leq j \leq n-1$$

Def: For L as above, the polynomial $p(r) := r^n + a_{n-1}r^{n-1} + \dots + a_2r^2 + a_1r + a_0$ is called the characteristic polynomial of L

Theorem 5 If the characteristic polynomial $p(r)$ of the n^{th} order const coeff linear operator L has n distinct roots $r_1, r_2, \dots, r_n$ (real roots) then a basis for $\ker(L)$ is

$$\{e^{r_1x}, e^{r_2x}, \dots, e^{r_nx}\}$$

i.e. $y_H = c_1e^{r_1x} + c_2e^{r_2x} + \dots + c_ne^{r_nx}$

proof: $L(e^{rx}) = p(r)e^{rx}$.

so if $p(r_i) = 0$, e^{r_ix} solves $L(y) = 0$

Since $\dim(\ker L) = n$, it suffices to show $\{e^{r_1x}, e^{r_2x}, \dots, e^{r_nx}\}$ are linearly ind! (since then they span, since there are n of them.)

Suppose $c_1e^{r_1x} + c_2e^{r_2x} + \dots + c_ne^{r_nx} \equiv 0$
take derivs!!

Did you prove in 2270, that the Vandermonde Determinant:

$$V = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ r_1 & r_2 & r_3 & \dots & r_n \\ r_1^2 & r_2^2 & r_3^2 & \dots & r_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} & r_2^{n-1} & r_3^{n-1} & \dots & r_n^{n-1} \end{vmatrix}$$

satisfies

$$V = \prod_{i < j} (r_j - r_i) \neq 0, \text{ if } r_1, r_2, \dots, r_n \text{ are distinct} \quad ??$$

There's another interesting way to prove $\{e^{r_1 x}, e^{r_2 x}, \dots, e^{r_n x}\}$ are linearly independent on $\mathbb{R}$:

$$\text{let } r_1 < r_2 < \dots < r_n$$

$$\text{let } c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_n e^{r_n x} \equiv 0 \quad x \in \mathbb{R}.$$

$$\div e^{r_n x}: \quad c_1 e^{(r_1 - r_n)x} + c_2 e^{(r_2 - r_n)x} + \dots + c_{n-1} e^{(r_{n-1} - r_n)x} + c_n \equiv 0$$

now take $\lim_{h \rightarrow \infty}$

$$\Rightarrow c_n = 0.$$

induct!



What about repeated roots?

example: Find the general soltn to

$$L(y) := y^{(4)} - 2y^{(2)} + y = 0$$

$$= (D^{(4)} - 2D^{(2)} + I)y \quad \text{operator notation, } D = \frac{d}{dx} \text{ is derivative operator}$$

$$= (D^2 - I)^2 y$$

$$= (D - I)^2 \circ (D + I)^2 y$$

↑
composition

Side computation

$$(D - r_1 I)e^{r_1 x} = r_1 e^{r_1 x} - r_1 e^{r_1 x} = 0$$

$$(D - r_1 I)(x e^{r_1 x})$$

$$= 1 e^{r_1 x} + r_1 x e^{r_1 x} - r_1 x e^{r_1 x}$$

$$\text{so } (D - r_1 I)^2 (x e^{r_1 x})$$

$$= (D - r_1 I) \circ (D - r_1 I)(x e^{r_1 x})$$

$$= (D - r_1 I)(e^{r_1 x})$$

$$= 0.$$

Theorem 6

$$\text{Let } L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y \quad a_j \text{ const}$$

So charact poly

$$p(r) = r^n + a_{n-1}r^{n-1} + \dots + a_1r + a_0$$

$$\text{and } L = D^{(n)} + a_{n-1}D^{(n-1)} + \dots + a_1D + a_0$$

(where D^k means $D \circ D \circ \dots \circ D$ k times)

$$\text{If } p(r) = (r-r_1)^k q(r) \quad (\text{degree } q_0 = n-k_0)$$

Then $\{e^{r_1x}, xe^{r_1x}, \dots, x^{k-1}e^{r_1x}\}$ are k lin. ind solutions to $L(y) = 0$.

$$\text{proof: } L(y) = p(D)(y)$$

$$= q(D) \circ (D-r_1I)^k(y)$$

$$= q(D) \circ [(D-r_1I)^k(y)]$$

$$\text{so } L(x^m e^{r_1x}) = q(D) \circ [(D-r_1I)^k(x^m e^{r_1x})]$$

$$= 0 \text{ for } m \leq k-1$$

$$\S \quad \frac{1}{2} \blacksquare$$

$$(D-r_1I)(e^{r_1x}) = r_1e^{r_1x} - r_1e^{r_1x} = 0$$

$$(D-r_1I)(xe^{r_1x}) = e^{r_1x} + r_1xe^{r_1x} - r_1xe^{r_1x} = e^{r_1x}; \quad (D-r_1I)^2(xe^{r_1x}) = 0.$$

$$(D-r_1I)x^m e^{r_1x} = m x^{m-1} e^{r_1x}$$

↑
one lower power

inductively,

$$(D-r_1I)^{m+1}(x^m e^{r_1x}) = 0$$

Notice $\{e^{r_1x}, xe^{r_1x}, \dots, x^{k-1}e^{r_1x}\}$ are lin ind on any interval, since

$$c_1e^{r_1x} + c_2xe^{r_1x} + \dots + c_{k-1}x^{k-1}e^{r_1x} \equiv 0$$

$$\text{iff } e^{r_1x} [c_1 + c_2x + \dots + c_{k-1}x^{k-1}] \equiv 0$$

$$\text{iff } c_1 + c_2x + \dots + c_{k-1}x^{k-1} \equiv 0$$

$$\text{iff } c_1 = c_2 = \dots = c_{k-1} = 0 \text{ since } \{1, x, x^2, \dots, x^{k-1}\} \text{ are lin ind. (why?)}$$

 $\blacksquare$

Theorem 7 If $p(r)$ factors,

$$p(r) = (r-r_1)^{k_1} (r-r_2)^{k_2} \dots (r-r_\ell)^{k_\ell} \quad r_1, r_2, \dots, r_\ell \text{ distinct}$$

then a basis for $L(y)=0$ sol's is

$$\left\{ e^{r_1 x}, x e^{r_1 x}, \dots, x^{k_1-1} e^{r_1 x}, e^{r_2 x}, x e^{r_2 x}, \dots, x^{k_2-1} e^{r_2 x}, \dots \right\}.$$

proof: From thm 2, the y_j all satisfy $L(y_j)=0$.

It is more work to show they are linearly independent (but true!)
I thought of a proof... can you?
(I use limits.)

fun

Question: Is it true that if $\{y_1, y_2, \dots, y_n\}$ solve $L(y)=0$, for

$$L(y) = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y \quad a_j \in C(I) \quad j=0, \dots, n.$$

and if

$$W = \det \begin{bmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{bmatrix} \neq 0 \text{ at } x=x_0, \text{ then } W(x) \neq 0 \quad \forall x \in I?$$