Math 2280-1
Friday Feb. 4

HW for Friday Feb. 11

3.1 1, 2, 4 5, 11, 13, 17 27, 29, 30, 31, 33, 34, 35
3.2 2 5 9 11 13 21 22 25 26
3.3 3, 10, 14 21 22, 29, 33, 37
3.4 4, 5, 6 13, 15, 19, 23

§3.1-3.2 Linear DE's and the structure of the solution space.

Recall, on Wednesday we discussed Theorems 1, 2, 3.
(We proved 1 & 3, and will come to understand why 2 is true, later)

<u>Theorem 1</u> $L(y) := a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y$ s.t. each a_j cont. on fixed interval I

then $L: C^n(I) \to C(I)$ is <u>linear</u>

$\{y(x) \mid y, y', \dots y^{(n)} \text{ are cont. on } I\}$

$\{z(x) \text{ s.t. } z \text{ cont on } I\}$.

<u>Theorem 2</u> $L(y) := y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y$, a_j cont on I $j = 0 \dots n-1$

$x_0 \in I$, $f \in C(I)$, $\vec{b} \in \mathbb{R}^n$

then $\exists!$ sol'n to IVP, defined on all of I, i.e. $y \in C^{(n)}(I)$.

$$\text{IVP} \begin{cases} L(y) = f \\ y(x_0) = b_0 \\ y'(x_0) = b_1 \\ \vdots \\ y^{(n-1)}(x_0) = b_{n-1} \end{cases}$$

– proof defered! –

<u>Theorem 3</u> $L: V \to W$ linear, $b \in W$.

then if $L(v_p) = b$, then the solution set to $L(v) = b$ is

$$\{v = v_p + v_H \text{ s.t. } L(v_H) = 0\}.$$
$$= \{v = v_p + v_H \text{ s.t. } v_H \in \ker(L)\}.$$

(page 4 Wed)

the "H" stands for "homogeneous sol'n", since $L(v) = 0$ is called the homogeneous equation

Exercise 1 : Use Theorems 1–3 to show why <u>every</u> solution to

$$y'' - 4y = 1 \qquad x \in \mathbb{R}$$

is of the form

$$y = -\frac{1}{4} + c_1 e^{2x} + c_2 e^{-2x}$$

(this was example 3 on Wed.)

hint: focus separately on y_P and y_H.

<u>Theorem 4</u> Let $L(y) = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_0(x)y$

$L : C^n(I) \to C(I)$ as in Theorem 2 on page 1

then $\ker(L) := \{ y \in C^n(I) \text{ s.t. } L(y) = 0 \}$ (also called nullspace(L))

has dimension n.

proof: Let $x_0 \in I$.

Consider the "initial value" transformation

$$T : \ker L \to \mathbb{R}^n$$

$$T(y) = \begin{bmatrix} y(x_0) \\ y'(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix}$$

- T is linear (check!)

- T is 1-1 and onto : this is precisely the existence and uniqueness theorem on page 1, applied in case the DE is $L(y) = 0$.

Therefore $\dim(\ker T) = 0$

$\dim(\underbrace{\text{Image}}_{\text{range}}(T)) = n$

So by the rank + nullity theorem from linear algebra

($\dim(\text{domain}) = \dim(\ker) + \dim(\text{Image})$)

$\dim(\ker L) = 0 + n = n$. (In fact, T is an isomorphism.) ∎

proof 2: (more concrete)

We'll use the $\exists!$ theorem to find a basis for solutions space to $L(y)=0$, and the basis will have n elements.

Let $y_j(x)$ be the soln to

$$\begin{cases} Ly_j = 0 \\ T(y_j) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} \leftarrow \text{entry } j \end{cases} = \vec{e}_j.$$

for $j=1,2,\cdots n$

We claim $\{y_1, y_2, \cdots y_n\}$ are a basis for $\ker L$.

Span Let $L(y)=0$ on I

Let $\begin{bmatrix} y(x_0) \\ y'(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix} = \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

Then $y(x)$ must actually equal

$$b_1 y_1 + b_2 y_2 + \cdots + b_n y_n$$

because this linear combo of $\{y_j\}$ has the same initial value vector as $y(x)$

check!

Independent: if $c_1 y_1 + c_2 y_2 + \cdots + c_n y_n \equiv 0$ ← the zero function on I

$\frac{d}{dx}$: $\Rightarrow$ $c_1 y_1' + c_2 y_2' + \cdots + c_n y_n' \equiv 0$

$\vdots$

$c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)} \cdots + c_n y_n^{(n-1)} \equiv 0$

this matrix is called the Wronskian matrix of $y_1, y_2, \cdots y_n$. It's determinant is called the Wronskian

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0} \qquad \forall x \in I$$

@ $x = x_0$, this reads $I\vec{c} = \vec{0} \longrightarrow \vec{c} = \vec{0}$ ∎

<u>Exercise 2</u> Find all solutions to

$$y''' + y'' - 30y' = 0$$

Hint: try for a basis of this 3-dim'l vector space made out of exponential functions, $y = e^{rx}$.

<u>Exercise 3</u> Find all sol'ns to

$$y''' + y'' - 30y' = 56e^x$$

Hint: try for a y_P of the form $y_P = Ce^x$.