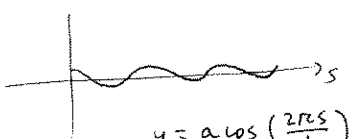
Math 2280-1
Wed. Feb. 23
§3.6

Hw for Friday Feb. 25
3.6 7, 13, 16, 20, 21, 22

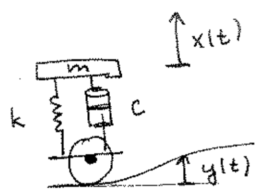
- resonance can occur in lots of interesting forced oscillation problems :

bumpy road p. 218:



$y = a\cos\left(\frac{2\pi s}{L}\right)$

road surface
period $= L$.



"car"

vertical acceleration:

$$m x'' = -k(x-y) - c(x'-y')$$

if $c = 0$

$$m x'' + kx = ky$$

if speed of car is v
then $s = vt$

$$= k a \cos\left(\frac{2\pi v}{L} t\right)$$

is a forced oscillation problem.
trouble when

$$\omega = \frac{2\pi v}{L} = \omega_0 = \sqrt{\frac{k}{m}}$$

$$v = \sqrt{\frac{k}{m}}\,\frac{L}{2\pi}$$

(freeway)
"washboard"

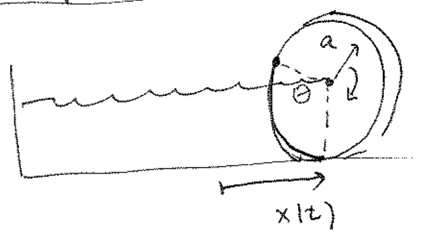
See also the front loading washing machine, #20 p. 222

Using total energy to deduce natural angular frequency:
(for conservative systems) KE + PE = const

1. $KE = \frac{1}{2}mv^2$ mass m, speed v (of center of mass)
2. $KE = \frac{1}{2}I\omega^2$ rotating object (ω = angular vel.) I = moment of intertia.
3. $PE = \frac{1}{2}kx^2$ stretched spring
4. $PE = mgh$ gravitational PE

total energy is additive

example 4 p217 M



$$E = \frac{1}{2}kx^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}Mx'(t)^2$$

$$x = a\theta$$
$$x' = a\theta' = a\omega$$

I for solid wheel $= \frac{1}{2}Ma^2$ (you can work this out!)

so

$$E = \frac{1}{2}kx^2 + \frac{1}{4}Ma^2\left(\frac{x'}{a}\right)^2 + \frac{1}{2}M(x')^2$$
$$= \frac{1}{2}kx^2 + \frac{3}{4}M(x')^2$$

$$0 = \frac{dE}{dt} = kxx' + \frac{3}{2}Mx'x'' = x'\left[kx + \frac{3}{2}Mx''\right]$$

$$\boxed{\frac{3}{2}Mx'' + kx = 0} \qquad \omega_0 = \sqrt{\frac{k}{\frac{3}{2}M}} = \sqrt{\frac{2k}{3M}}$$

moment of inertia for rotating disk.

$$KE = \iint \frac{1}{2}\underbrace{(r\omega)^2}_{v^2}\underbrace{\rho\, r\,dr\,d\theta}_{dm} \qquad \rho = \frac{M}{\pi a^2}$$

$$= \frac{1}{2}\rho\omega^2 \underbrace{\int_0^{2\pi}\underbrace{\int_0^a r^3\,dr}_{\frac{a^4}{4}}\,d\theta}_{\frac{\pi a^4}{2}}$$

$$= \cancel{\frac{1}{4}\pi\frac{M}{\pi a^2}}$$

$$= (\rho\pi a^2)\,\frac{1}{4}(a\omega)^2$$

$$= \frac{1}{4}Ma^2\omega^2$$

$$= \frac{1}{2}I\omega^2 \quad \text{for } \boxed{I = \frac{1}{2}Ma^2.}$$

Case 3 Forced oscillations with damping

this differential equation might look familiar...

$$\begin{cases} x''(t) + 2x'(t) + 5x(t) = 10\cos t \\ x(0) = 1 \\ x'(0) = 2 \end{cases}$$

$x_H(t)$

$x_p(t)$:

IVP:

question to fool you: what form would your guess for $x_p(t)$ take if RHS was $10\cos 2t$?

general case: (for reference)

$$m x'' + c x' + k x = F_0 \cos\omega t \qquad c \neq 0$$

$$
\begin{aligned}
k\,(\quad & x_p = A\cos\omega t + B\sin\omega t \\
+\ c\,(\quad & x_p' = -A\omega\sin\omega t + B\omega\cos\omega t \\
+\ m\,(\quad & x_p'' = -A\omega^2\cos\omega t - B\omega^2\sin\omega t
\end{aligned}
$$

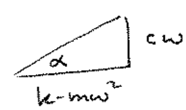
$$
\begin{aligned}
L(x_p) = \cos\omega t \left[A(k-m\omega^2) + Bc\omega \right] \qquad &= \cos\omega t\,[F_0] \\
+ \sin\omega t \left[A(-c\omega) + B(k-m\omega^2) \right] \qquad &+ \sin\omega t\,[0]
\end{aligned}
$$

$$\begin{bmatrix} k-m\omega^2 & c\omega \\ -c\omega & k-m\omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{(k-m\omega^2)^2 + c^2\omega^2} \begin{bmatrix} k-m\omega^2 & -c\omega \\ c\omega & k-m\omega^2 \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{F_0}{(k-m\omega^2)^2 + c^2\omega^2} \begin{bmatrix} k-m\omega^2 \\ c\omega \end{bmatrix}$$

so $\quad x_p(t) = C\cos(\omega t - \alpha)$



$$C = \sqrt{A^2+B^2}$$

$$C = \frac{F_0}{(k-m\omega^2)^2 + c^2\omega^2} \sqrt{(k-m\omega^2)^2 + c^2\omega^2}$$

$$\boxed{C = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + c^2\omega^2}} \ , \quad \alpha = \arccos\left(\frac{k-m\omega^2}{C}\right)}$$

notice denom can be very small if
$\omega \approx \omega_0$ (makes 1st term small)
$c \approx 0$ (makes second term small).
} this makes $C \gg F_0$ "practical resonance"

then

$$x(t) = \underbrace{C\cos(\omega t - \alpha)}_{x_{sp}(t)} + \underbrace{x_H(t)}_{x_{tr}(t)}$$

$$x_{tr}(t) = \begin{cases} c_1 e^{-r_1 t} + c_2 e^{-r_2 t} & \text{overdamped} \\ e^{-rt}(c_1 + c_2 t) & \text{crit damped} \\ e^{-at}(A\cos bt + B\sin bt) & \text{underdamped.} \end{cases}$$

practical resonance occurs when C is $\gg F_0$, ← you decide.

i.e. when $\omega \approx \omega_0$, $c \approx 0$.

example :

$$x'' + 2x' + 26x = F_0 \cos \omega t$$

plot the fcn

$$\frac{1}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}, \text{ as a fcn of } \omega \quad \left(\begin{array}{l} m = 1 \\ k = 26 \\ c = 2 \end{array} \right.$$

$$= \frac{1}{\sqrt{(26-\omega^2)^2 + 4\omega^2}}$$

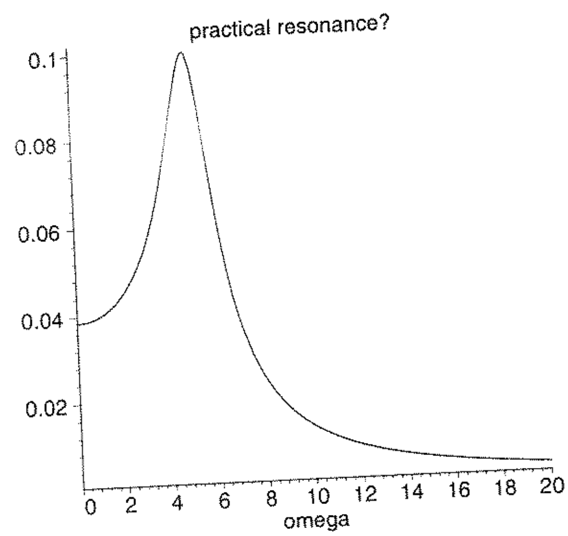


figure 3.6.9 p. 221

$\omega_0 = \sqrt{26} \approx 5$