Math 2280-1
Tuesday Feb. 15

§3.5 Finding y_p for $Ly = f$

$$L = D^{(n)} + a_{n-1}D^{(n-1)} + \dots + a_1 D + a_0 I$$

Method 1 Undetermined coefficients usually in case all a_j's are constants.

Theorem 1 If $f \in V$ where V is finite dimensional and if $L: V \to V$ is 1-1 (i.e. $N(L|_V) = \{0\}$), then $\exists!\ y_p \in V$ with $Ly_p = f$

proof: by $\underbrace{\dim(N(L))}_{=0} + \dim(\text{Image}(L)) = \dim V$

we deduce L is 1-1 and onto (an isomorphism) ∎

If V has a basis
$\{g_1, g_2, \dots g_k\}$
then finding $y_p = c_1 g_1 + c_2 g_2 + \dots + c_k g_k$ is called the method of undetermined coefficients because you need to find $c_1, \dots c_k$.

Examples

$V = \text{span}\{e^{ax}\}$ — a not a root of $p(r) = r^n + a_{n-1}r^{n-1} + \dots + a_1 r + a_0$.

$V = \text{span}\{e^{ax}\cos bx, e^{ax}\sin bx\}$ — $r = a \pm bi$ not a root of $p(r)$

$V = \text{span}\{1, x, \dots x^k\}$ — $r = 0$ not a root of $p(r)$

$V = \text{span}\{e^{ax}, xe^{ax}, \dots x^k e^{ax}\}$ — a not a root of $p(r)$

$V = \text{span}\{e^{ax}\cos bx, e^{ax}\sin bx, xe^{ax}\cos bx, xe^{ax}\sin bx, \dots x^k e^{ax}\cos bx, x^k e^{ax}\sin bx\}$

$r = a \pm bi$ not a root of $p(r)$

(we did typical examples yesterday).

Notice in all cases, $\begin{matrix} D: V \to V \\ I: V \to V \end{matrix} \Rightarrow L: V \to V$ the conditions on a, $a \pm bi$, force L to be an isomorphism

Theorem 2 (Rule 2, p. 205).

$L = D^{(n)} + a_{n-1}D^{(n-1)} + \dots + a_1 D + a_0 I$ $\quad a_j$ const.

$f \in V$, one of the page 1 subspaces.

if $r_1 = a+bi$ is a root of $p(r)$, $p(r) = q(r)(r-r_1)^s$ with $q(r_1) \neq 0$

(so s is the algebraic multiplicity of r_1 in $p(r)$)

Let $x^s V := \{ x^s g(x) \text{ s.t. } g \in V \}$.

Then $L : x^s V \to V$ is a bijection so $\exists!\ y_p \in x^s V$ s.t. $L(y_p) = f$.

proof: $L = q(D) \circ (D - r_1 I)^s$

$(D - r_1 I)^s : x^s V \longrightarrow V$ because each application of $D - r_1 I$ sends $x^\ell V \to x^{\ell-1} V$.

furthermore, this map is injective, since $(D-r_1 I)^s$ lowers powers of x by exactly s.

So

$$x^s V \xrightarrow[\text{iso}]{(D-r_1 I)^s} V \xrightarrow[\text{isomorphism}]{q(D)} V \text{ is isomorphism.}$$

∎

Example (p. 4 Monday)

$L(y) = y'' + 2y' + y$

Solve $L(y) = e^{-x}$

notice $p(r) = (r+1)^2$

There's a method that's guaranteed to find a y_P for $L(y_P) = f$, if you've already found y_H – it looks magic now but will be motivated better later in course.

__variation of parameters__ : If, for $L(y) := y^{(n)} + a_{n-1}(x) y^{(n-1)} + \dots + a_1(x) y' + a_0(x) y$

you already have a basis for $\ker(L)$, i.e. the sol'n set to $L(y) = 0$, there is a formula to find a particular sol'n y_P to

$$L(y_P) = f.$$

(this will work for any right hand side, not just the special ones we considered for "undetermined coeff".)

derivation: Let $\{y_1, y_2, \dots y_n\}$ a basis for $\ker(L)$

we search for y_P expressed as

$u_j = u_j(x)$ fcns! (hence the name of the method).

$$
\begin{aligned}
a_0\,(\quad & y_P(x) = u_1 y_1 + u_2 y_2 + \dots + u_n y_n \\
a_1\,(\quad & y_P' = u_1 y_1' + u_2 y_2' + \dots + u_n y_n' + \underbrace{u_1' y_1 + u_2' y_2 + \dots + u_n' y_n}_{\text{set } \equiv 0} \\
+\ a_2\,(\quad & y_P'' = u_1 y_1'' + u_2 y_2'' + \dots + u_n y_n'' + \underbrace{u_1' y_1' + u_2' y_2' + \dots + u_n' y_n'}_{\text{set } \equiv 0} \\
& \vdots \\
a_{n-1}\,(\quad & \\
1\,(\quad & y_P^{(n)} = u_1 y_1^{(n)} + u_2 y_2^{(n)} + \dots + u_n y_n^{(n)} + \underbrace{u_1' y_1^{(n-1)} + \dots + u_n' y_n^{(n-1)}}_{\text{set } \equiv f}
\end{aligned}
$$

$$\Rightarrow \quad L(y_P) = \underbrace{u_1 L(y_1) + \dots + u_n L(y_n)}_{=0!} + f$$

conditions in matrix form

$$
\underbrace{\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{bmatrix}}_{W(y_1, \dots y_n).}
\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f \end{bmatrix}
$$

$$
\boxed{\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = [W]^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f \end{bmatrix}}
$$

then find choices of each $u_j(x)$ by antidifferentiation,

& get

$$y_P = u_1 y_1 + \dots + u_n y_n \ !$$

__Use Theorem:__ since $\{y_1, \dots y_n\}$ is a basis we can solve all IVP's at each $x \Rightarrow [W]$ has rank n __at each__ $x \Rightarrow W^{-1}$ exists at each x!

Example use variation of parameters to find a particular soltn to

$$y'' + 2y' + y = e^{-x}$$

(we did this via undetermined coeffs)

2280-1
Review sheet
for exam 1.

Be able to explain
reasoning & concepts
as well as work problems

1^{st} order ODE's. Chapters 1.1-1.5, 2.1-2.3 (not 1.6, 2.4-2.6)

- Recognize and solve DE's and IVP's for
  - $\frac{dy}{dx} = f(x)$ §1.2 antidiff
  - $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ §1.4 separable
  - $\frac{dy}{dx} + P(x)y = Q(x)$ §1.5 linear
- slope fields, $\exists !$ for IVP §1.3, 2.2
  phase portraits.
  - geometric meaning of $\frac{dy}{dx} = f(x,y)$ for the graph $y=y(x)$
  - sketch slope fields using isoclines
  - $\exists !$ thm for IVP
  - autonomous DE's $\frac{dy}{dx} = f(y)$
    - equilibrium sol'ns
    - phase portraits
    - stability
- applications & modeling
  - separable
    - exp growth & decay
    - Newton's law of cooling
    - Toricelli
    - logistic / doomsday ext/harvesting (§2.1)
    - acceleration with drag terms (§2.3)
  - linear
    - mainly mixing problems (§1.5)
    - also some of §2.3 models
    - if $P(x)$ is const, i.e.
      $$y' + a_0 y = f$$
      this theory connects to chapter 3 $y = y_P + y_H \cdots$
      use of e^{rt}, etc.

Higher order linear DE's (3.1-3.5)

- Why $L(y) := y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y$
  is called <u>linear</u>
  Why the general sol'n to $L(y) = f$ is $y = y_P + y_H$; more general superposition
  Dimension of sol'n space to $L(y) = 0$
  (and its relationship to $\exists !$ thm for IVP)
  linear ind & dep of fcns
  Wronskian matrix & determinant
- Solving $L(y) = 0$ if L has constant coeffs a_j
  e^{rx}, $p(r)$, real roots (distinct, repeated), complex roots (distinct & repeated)
  using complex exponentials, Euler, cos & sin addition angle formulas
- Mechanical vibrations
  - pendulum & spring models
  - simple harmonic motion (amp, phase, ang. freq, freq, period, etc. ABC triangle)
  - the three kinds of damping, and corresponding solution types.
- Finding particular sol'ns to $Ly = f$, using $y = y_P + y_H$ to solve IVP's.