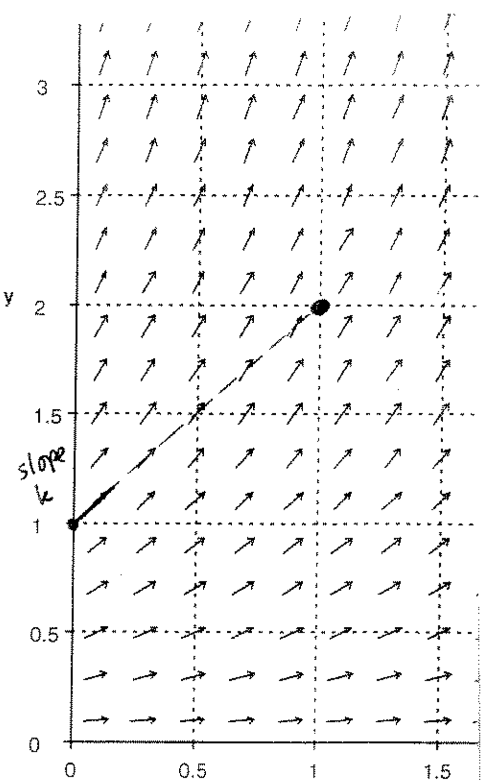
Numerical approximations to DE solutions

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

approximate soltns on interval $[x_0, x_n]$ with n subintervals of length $h = \frac{x_n - x_0}{n}$.

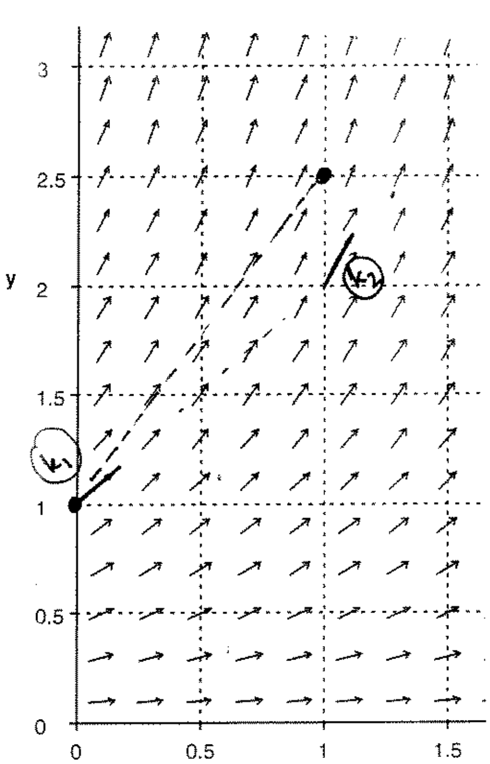
Euler	Improved Euler	Runge Kutta
$k = f(x_j, y_j)$	$k_1 = f(x_j, y_j)$	$k_1 = f(x_j, y_j)$
$x_{j+1} = x_j + h$	$k_2 = f(x_j + h, y_j + hk_1)$	$k_2 = f(x_j + \frac{h}{2}, y_j + \frac{h}{2} k_1)$
$y_{j+1} = y_j + hk$	$k = \frac{1}{2}(k_1 + k_2)$	$k_3 = f(x_j + \frac{h}{2}, y_j + \frac{h}{2} k_2)$
	$x_{j+1} = x_j + h$	$k_4 = f(x_j + h, y_j + hk_3)$
	$y_{j+1} = y_j + hk$	$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
		$x_{j+1} = x_j + h$
		$y_{j+1} = y_j + hk$

illustrated for $\begin{cases} y' = y \\ y(0) = 1 \end{cases}$ $n = 1$, interval $[0, 1]$.



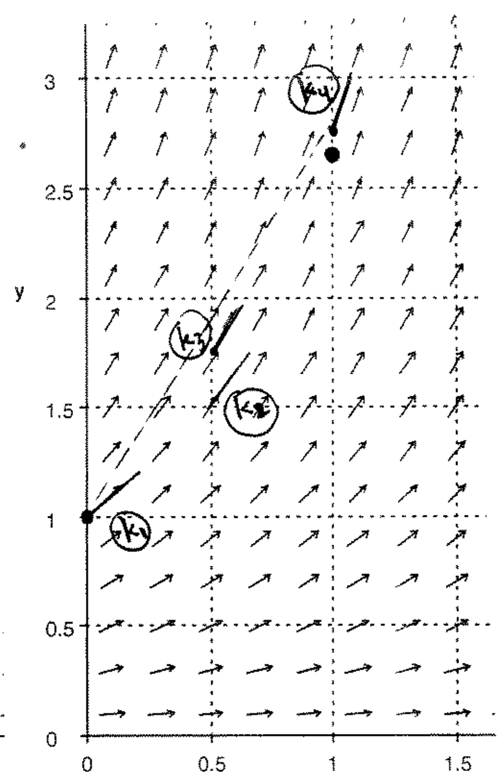
$x_0 = 0$
$y_0 = 1$
$k = 1$
$x_1 = 1$
$y_1 = 1 + 1 \cdot 1 = 2$
est. for e

Euler



$k_1 = 1$
$k_2 = 2$
$k = \frac{1}{2}(k_1 + k_2) = 1.5$
$y_1 = 1 + (1.5)$
$= 2.5$

Improved Euler



$k_1 = 1$
$k_2 = 1.5$
$k_3 = 1.75$
$k_4 = 2.75$
$k = \frac{1}{6}(1 + 2k_1 + 2k_2 + k_3)$
$\cong 1.708$
$\boxed{y_1 \cong 2.708}$

Runge Kutta!