Name.......Solutions...

I.D. number...

Math 2280-1
Second Midterm
April 1, 2011

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible, and the point values for each problem are indicated in the right-hand margin.
Good Luck!

	Score	possible
1	______	(30)
2	______	(35)
3	______	(25)
4	______	(10)
total	______	(100)

1a) Let A be the matrix

$$A = \begin{bmatrix} -.02 & 0 \\ .02 & -.04 \end{bmatrix}.$$

Find the eigenvalues of A, and corresponding eigenvectors. Work carefully, because you'll be using this information in the rest of problem 1.

(10 points)

since A is lower triangular the eigenvalues are $-.02$ and $-.04$.

By looking at the 2nd column we see that for $\lambda = -.04$, $\vec{v} = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$\lambda = -.02$

$$A - -.02I \left|\begin{matrix} 0 \\ 0 \end{matrix}\right. \rightarrow \begin{matrix} 0 & 0 \\ .02 & -.02 \end{matrix} \left|\begin{matrix} 0 \\ 0 \end{matrix}\right. \qquad \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$\begin{matrix} 1 & -1 \\ 0 & 0 \end{matrix} \left|\begin{matrix} 0 \\ 0 \end{matrix}\right.$$

$$\boxed{\begin{matrix} \lambda = -.02 & \lambda = -.04 \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{matrix}}$$

1b) Compute the matrix exponential e^{At} for this matrix A.

(10 points).

$$\text{a FMS} = \begin{bmatrix} e^{-.02t} & 0 \\ e^{-.02t} & e^{-.04t} \end{bmatrix}$$

$$\begin{matrix} \uparrow & \\ e^{\lambda_1 t}\vec{v}_1 & e^{\lambda_2 t}\vec{v}_2 \end{matrix}$$

so $e^{At} = \Phi(t)\Phi(0)^{-1}$

$$\Phi(0) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\Phi(0)^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{-.02t} & 0 \\ e^{-.02t} & e^{-.04t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\boxed{e^{At} = \begin{bmatrix} e^{-.02t} & 0 \\ e^{-.02t} - e^{-.04t} & e^{-.04t} \end{bmatrix}}$$

1c) Now consider the initial value problem

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -.02 & 0 \\ .02 & -.04 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \end{bmatrix}$$

Solve this IVP using your work from page 1. (By the way, this is exactly the initial value problem that arose from the pollution incident at Lakes Alice and Bob on exam 1, before we knew how to systematically solve linear systems of differential equations.)

(10 points)

$\vec{x}(t) = e^{At}\vec{x}_0$ is soln to $\begin{cases} \vec{x}' = A\vec{x} \\ \vec{x}(0) = \vec{x}_0 \end{cases}$ (just like for scalar exponential growth/decay problems)

so $\vec{x}(t) = e^{At}\begin{bmatrix} 50 \\ 0 \end{bmatrix}$ = 50 times 1st col. of e^{At}

$$\boxed{\vec{x}(t) = 50 \begin{bmatrix} e^{-.02t} \\ e^{-.02t} - e^{-.04t} \end{bmatrix}}$$

2) We used the principle of conservation of energy and the method of linearization to derive the second order linear differential equation which describes undamped unforced pendulum motion. This differential equation is

$$L\cdot\theta''(t) + g\cdot\theta(t) = 0$$

where $\theta(t)$ is the angle from vertical, L is the length of the (massless) rod or string, m is the mass at the end of the string, and $g = 9.8\ \frac{m}{s^2}$ is the acceleration of gravity on earth.

2a) Derive the linear model above, using the fact that kinetic plus potential energy is constant for any solution to this conservative system. This will yield a non-linear differential equation which you can linearize to the one above, under the assumption that $\theta(t)$ is near zero.

(15 points)

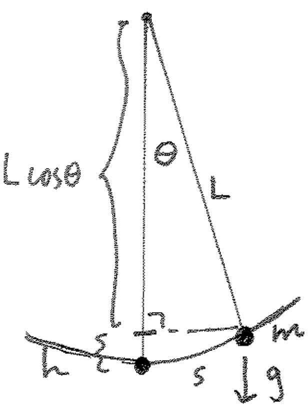
$$TE = KE + PE$$
$$= \tfrac{1}{2}mv^2 + mgh$$

h = ht, say from equil. ht when $\theta = 0$

$$TE = \tfrac{1}{2}mL^2(\theta')^2 + mg(L - L\cos\theta)$$

$$0 = \frac{d}{dt}(TE) = \cancel{\tfrac{1}{2}}mL^2\cancel{2}\theta'\theta'' - mgL(-\sin\theta)\theta'(t)$$

$$0 = mL\theta'(L\theta'' + g\sin\theta)$$



So on any time interval for which $\theta' \neq 0$, must have $\boxed{L\theta'' + g\sin\theta = 0}$

$s = L\theta$

$s' = L\theta'(t)$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

so for $|\theta|$ small, $\sin\theta \approx \theta$, $\boxed{L\theta'' + g\theta = 0}$

(will hold $\forall t$, by continuity)

2b) If you wished to create a pendulum for which the natural period was 2 seconds/cycle, what length L would you choose? (A symbolic answer suffices - the correct decimal value for L is close to 1 meter.)

(5 points)

$$\theta'' + \frac{g}{L}\theta = 0$$

$$w_0^2 = \frac{g}{L}$$

$$w_0 = \sqrt{\frac{g}{L}}$$

$$T = 2 = \frac{2\pi}{w_0} \Rightarrow w_0 = \pi$$

$$\pi = \sqrt{\frac{g}{L}}$$

$$\pi^2 = \frac{g}{L}$$

$$\boxed{L = \frac{g}{\pi^2}} \approx .99 \text{ meters.}$$

2c) Suppose that $\frac{g}{L} = 4$, so that the pendulum differential equation for $\theta(t)$ reduces to

$$(1) \qquad \theta''(t) + 4\cdot\theta(t) = 0.$$

Explain how solutions to this second order differential equation are related to solutions to the first order system:

$$(2) \qquad \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad \begin{aligned} x_1' &= x_2 \\ x_2' &= -4x_1 \end{aligned}$$

(5 points)

If $\theta(t)$ solves (1) then $\begin{bmatrix} \theta \\ \theta' \end{bmatrix}$ solves (2)

If $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ solves (2) then $x_1(t)$ solves (1).

2d) Using whatever method you prefer, compute e^{At} for the matrix above,

$$A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}.$$

(10 points)

soltn basis to (1) is $\{\cos 2t, \sin 2t\}$

yields two soltns $\begin{bmatrix} \theta(t) \\ \theta'(t) \end{bmatrix}$ to (2)

$$\Rightarrow \quad FMS = \begin{bmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{bmatrix}$$

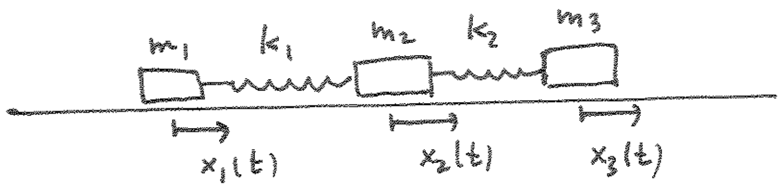
note, if I replace the second colmn with $\frac{1}{2}$ of it, I get

$$\boxed{e^{At} = \begin{bmatrix} \cos 2t & \frac{1}{2}\sin 2t \\ -2\sin 2t & \cos 2t \end{bmatrix}}$$ since @ $t=0$ this matrix is I

(alternately, $e^{At} = \Phi(t)\Phi(0)^{-1}$).

(It is also straight forward to compute this e^{At} with $\lambda, \vec{v}$ method, or with power series)

3) Consider the following configuration of 3 masses held together with two springs (a 3 car train), with positive displacements from equilibrium for each mass measured to the right, as usual.



3a) Use Newton's Law and Hooke's usual linearization to derive the system of 3 second order differential equations governing the masses' motion.

(5 points)

$$m_1 x_1'' = k_1(x_2 - x_1) \qquad = -k_1 x_1 + k_2 x_2$$
$$m_2 x_2'' = -k_1(x_2 - x_1) + k_2(x_3 - x_2) \qquad = k_1 x_1 - 2k_2 x_2 + k_2 x_3$$
$$m_3 x_3'' = -k_2(x_3 - x_2) \qquad = k_2 x_2 - k_2 x_3$$

3b) What is the dimension of the solution space to this problem? Explain.

(5 points)

6 (equivalent to a system of 6 1^{st} order linear homog. DE's, for the fcns x_1, x_1', x_2, x_2', x_3, x_3')

3c) Assume that all three masses are identical, and the two spring constants are also equal. Assume further that units have been chosen so that the numerical value "m" of each mass equals the numerical value "k" of each Hooke's constant. Show that in this case the system in (3a) reduces to

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \\ x_3''(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

(5 points)

$m_i = m$

$k_i = k = m$

from 3a) (divide by m):

$$x_1'' = -x_1 + x_2$$
$$x_2'' = x_1 - 2x_2 + x_3$$
$$x_3'' = x_2 - x_3$$

$\lambda = -1$
$\omega = \sqrt{-\lambda} = 1$

$\lambda = 0$
get $(c_1 + c_2 t)\vec{v}$

$\lambda = -3,\ \omega = \sqrt{3}$

3d) Maple says that for the matrix in this second order system,

> *Eigenvectors*(*A*);

$$\begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix} \qquad \textbf{(1)}$$

(Recall, the first vector is the list of eigenvalues, and the matrix has corresponding eigenvectors in its columns.)
Use this information to write down the general solution to the system of differential equations in (3c).
Describe the three fundamental mode oscillations. (Technically, one of them isn't actually an oscillation).
(10 points)

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \underbrace{(c_1 + c_2 t)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{} + \underbrace{\underset{C\cos(t-\alpha)}{(c_3 \cos t + c_4 \sin t)}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{} + \underbrace{\underset{D\cos(\sqrt{3}t-\beta)}{(c_4 \cos\sqrt{3}t + c_5 \sin\sqrt{3}t)}}_{}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

each car moving at constant velocity c_2 (or, if $c_2 = 0$, cars at rest at location c_1)

cars 1 & 3 out of phase with equal amplitude. car 2 at rest

cars 1 & 3 in phase, out of phase with car 2 which oscillates with twice the amplitude.

4) Let A be a constant matrix, $f(t)$ a continuous vector-valued function on an interval containing the origin, and consider non-homogeneous the initial value problem

$$x'(t) - Ax = f(t)$$
$$x(0) = x_0$$

for the vector-valued function $x(t)$. Use matrix exponentials to derive a formula for the solution to this IVP, as we discussed in class and the text discusses at the end of Chapter 5.

(10 points)

$$x' - Ax = f$$

these are equal because of product rule and because $e^{-At} = e^{-At}A$

$$e^{-At}(x' - Ax) = e^{-At} f$$

$$\frac{d}{dt}(e^{-At} x) = e^{-At} f$$

$\int_0^t$:

$$e^{-At} x(t) - x_0 = \int_0^t e^{-As} f(s)\,ds$$

$$e^{-At} x(t) = x_0 + \int_0^t e^{-As} f(s)\,ds$$

e^{At}:

$$\boxed{x(t) = e^{At} x_0 + e^{At}\int_0^t e^{-As} f(s)\,ds}$$

$$e^{At} e^{-At} = e^{At - At} = e^{[0]} = I$$

because At & $-At$ commute