

Name.....

I.D. number.....

Math 2280-1
Second Midterm
April 1, 2011

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work and justify your conclusions.** There are 100 points possible, and the point values for each problem are indicated in the right-hand margin.
Good Luck!

Score possible

1 _____ (30)

2 _____ (35)

3 _____ (25)

4 _____ (10)

total _____ (100)

1a) Let A be the matrix

$$A = \begin{bmatrix} -.02 & 0 \\ .02 & -.04 \end{bmatrix}.$$

Find the eigenvalues of A , and corresponding eigenvectors. Work carefully, because you'll be using this information in the rest of problem 1.

(10 points)

1b) Compute the matrix exponential e^{At} for this matrix A .

(10 points).

1c) Now consider the initial value problem

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -.02 & 0 \\ .02 & -.04 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$
$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \end{bmatrix}$$

Solve this IVP using your work from page 1. (By the way, this is exactly the initial value problem that arose from the pollution incident at Lakes Alice and Bob on exam 1, before we knew how to systematically solve linear systems of differential equations.)

(10 points)

2) We used the principle of conservation of energy and the method of linearization to derive the second order linear differential equation which describes undamped unforced pendulum motion. This differential equation is

$$L \cdot \theta''(t) + g \cdot \theta(t) = 0$$

where $\theta(t)$ is the angle from vertical, L is the length of the (massless) rod or string, m is the mass at the end of the string, and $g = 9.8 \frac{m}{s^2}$ is the acceleration of gravity on earth.

2a) Derive the linear model above, using the fact that kinetic plus potential energy is constant for any solution to this conservative system. This will yield a non-linear differential equation which you can linearize to the one above, under the assumption that $\theta(t)$ is near zero.

(15 points)

2b) If you wished to create a pendulum for which the natural period was 2 seconds/cycle, what length L would you choose? (A symbolic answer suffices - the correct decimal value for L is close to 1 meter.)

(5 points)

2c) Suppose that $\frac{g}{L} = 4$, so that the pendulum differential equation for $\theta(t)$ reduces to

$$\theta''(t) + 4\theta(t) = 0.$$

Explain how solutions to this second order differential equation are related to solutions to the first order system:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

(5 points)

2d) Using whatever method you prefer, compute e^{At} for the matrix above,

$$A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}.$$

(10 points)

3) Consider the following configuration of 3 masses held together with two springs (a 3 car train), with positive displacements from equilibrium for each mass measured to the right, as usual.

3a) Use Newton's Law and Hooke's usual linearization to derive the system of 3 second order differential equations governing the masses' motion.

(5 points)

3b) What is the dimension of the solution space to this problem? Explain.

(5 points)

3c) Assume that all three masses are identical, and the two spring constants are also equal. Assume further that units have been chosen so that the numerical value "m" of each mass equals the numerical value "k" of each Hooke's constant. Show that in this case the system in (3a) reduces to

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \\ x_3''(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

(5 points)

3d) Maple says that for the matrix in this second order system,

$\left[\begin{array}{l} > \text{Eigenvectors}(A); \end{array} \right.$

$$\left[\begin{array}{l} -1 \\ 0 \\ -3 \end{array} \right], \left[\begin{array}{ccc} -1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & 1 \end{array} \right]$$

(1)

(Recall, the first vector is the list of eigenvalues, and the matrix has corresponding eigenvectors in its columns.)

Use this information to write down the general solution to the system of differential equations in (3c).

Describe the three fundamental mode oscillations. (Technically, one of them isn't actually an oscillation).

(10 points)

4) Let A be a constant matrix, $f(t)$ a continuous vector-valued function on an interval containing the origin, and consider non-homogeneous the initial value problem

$$\begin{aligned}x'(t) - Ax &= f(t) \\ x(0) &= x_0\end{aligned}$$

for the vector-valued function $x(t)$. Use matrix exponentials to derive a formula for the solution to this IVP, as we discussed in class and the text discusses at the end of Chapter 5.

(10 points)