Name Solutions

Student I.D. ______________________

Math 2280–1
Exam #1
February 18, 2011

Please show all work for full credit. This exam is closed book and closed note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right–hand margin. **Good Luck!**

Score	POSSIBLE
1______	35
2______	15
3______	10
4______	35
5______	5
TOTAL______	100

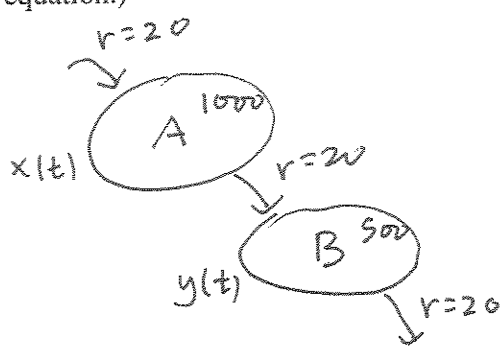
1) There has been a pollution accident! A dump truck spilled 50 tons of a water soluable pollutant into Lake Alice. This usually pristine lake has a constant volume of 1000 acre–feet. (An acre–foot is the amount of water in a volume of cross–sectional area one acre, and depth one foot.) Two tributary streams feed pure mountain spring water into Lake Alice at a combined rate of 20 acre–feet per day. A single stream flows out of Lake Alice at a rate of 20 acre–feet per day, maintaining Alice's constant volume. This outlet stream flows directly into Lake Ben, located nearby. Lake Ben has no other tributaries, and its outlet stream also flows at 20 acre–feet per day, keeping Lake Ben's volume at 500 acre–feet.

1a) Use the information above, your differential equations modeling ability, and the assumption that the pollutant salt in each lake quickly mixes to uniform concentrations, to show that the amount of pollutant in Lake Alice t days after the spill is given by

$$x(t) = 50 \cdot e^{-0.02 \cdot t} \text{ tons}$$

(An input–output diagram might help get you started. The correct differential equation is one of our basic ones, and you can quote the solution rather than rederive it, once you identify the differential equation.)

(10 points)



$$\frac{dx}{dt} = r_i c_i - r_o c_o$$

$$= 0 - 20\frac{x}{1000} = -.02x$$

$$\boxed{\begin{cases} \dfrac{dx}{dt} = -.02x \\ x(0) = 50 \end{cases}}$$

exponential decay so $x(t) = x_0 e^{-.02t}$

$$\boxed{x(t) = 50e^{-.02t} \text{ tons}}$$

1b) Using the solution $x(t)$ from part (1a) and the other information in this problem, derive the differential equation governing that the amount of pollutant $y(t)$ in Lake Ben t days after the accident, namely:

$$\frac{dy}{dt} = -0.04 \cdot y(t) + e^{-0.02 \cdot t}.$$

(10 points)

$$\frac{dy}{dt} = r_i c_i - r_o c_o$$

$$= 20\frac{x}{1000} - 20\frac{y}{500}$$

$$= \frac{20 \cdot 50e^{-.02t}}{1000} - .04y$$

$$\boxed{\frac{dy}{dt} = e^{-.02t} - .04y}$$

1c) Lake Ben is initially unpolluted. Find a formula for the amount of pollutant $y(t)$ in lake Ben, t days after the spill in upstream Lake Alice. For your reference, here is the differential equation from page 1:

$$\begin{cases} \frac{dy}{dt} = -0.04 \cdot y(t) + e^{-0.02 \cdot t}. \\ y(0) = 0 \end{cases}$$

(10 points)

$$y' + .04y = e^{-.02t}$$

$$e^{.04t}(y' + .04y) = e^{.04t} e^{-.02t} = e^{.02t}$$

$$(e^{.04t} y)' = e^{.02t}$$

$$e^{.04t} y = \int e^{.02t} dt = \frac{1}{.02} e^{.02t} + C$$

$$= 50 e^{.02t} + C$$

$$\div e^{.04t}: \quad y = 50 e^{-.02t} + Ce^{-.04t}$$

$$@ \begin{matrix} t=0 \\ y=0 \end{matrix} \text{ so } \quad 0 = 50 + C, \text{ so } C = -50$$

$$\boxed{y(t) = 50e^{-.02t} - 50e^{-.04t}}$$

1d) How many days after the spill is the amount of pollutant in Lake Ben at a maximum? (Your answer may involve logarithms....if you want a decimal approximation to see if your answer is sensible, and you didn't bring a scientific calculator, it may help to know that $\ln(2) = 0.69$, to two decimal places.)

(5 points)

$y(0) = 0$

$y(t) > 0$ for $t > 0$

$y(t) \to 0$ as $t \to \infty$

so y has a pos. max, @ t s.t. $y'(t) = 0$.

$$y'(t) = 50(-.02)e^{-.02t} - 50(-.04)e^{-.04t}$$

$$0 = -e^{-.02t} + 2e^{-.04t}$$

$$e^{-.02t} = 2e^{-.04t}$$

$e^{.04t}$ ·

$$e^{.02t} = 2$$

$$.02t = \ln 2$$

$$\boxed{t = \frac{\ln 2}{.02} = 50 \ln 2 \approx 35 \text{ days}}$$

2) Consider a body that moves horizontally through a medium with positive velocity, with drag resistance proportional to the *square* of the velocity and no other forces, so that

$$\frac{dv}{dt} = -k \cdot v^2.$$

For example, this model might describe the deceleration of a boat in water, after the engine is turned off and the boat continues to coast along in a straight line.

2a) Let v_0 be the boat's initial velocity, at time $t = 0$. Derive a formula for the boat's velocity function $v(t)$.

(10 points)

$$-\frac{dv}{v^2} = +k\,dt$$

integrate: $\frac{1}{v} = +kt + C$

@ $t_0 = 0$ $\quad \frac{1}{v_0} = C$

$$\frac{1}{v} = +kt + \frac{1}{v_0} = \frac{+ktv_0 + 1}{v_0}$$

$$\boxed{v = \frac{v_0}{1 + ktv_0}}$$

2b) Let x_0 be the boat's initial position. Derive a formula for its position function $x(t)$.

(5 points)

$$\frac{dx}{dt} = \frac{v_0}{1 + ktv_0}$$

$$x = \int \frac{v_0}{1 + ktv_0}\,dt = \frac{1}{k}\ln(1 + ktv_0) + C$$

by inspection
or substitution

$u = 1 + ktv_0$

$du = kv_0\,dt$

$$\int \frac{1}{u}\frac{1}{k}\,du = \frac{1}{k}\ln|u| = \frac{1}{k}\ln u \text{ if } u > 0.$$

@ $t = 0$ $\quad x_0 = 0 + C$ because $\ln 1 = 0$

so $\boxed{x = x_0 + \frac{1}{k}\ln(1 + ktv_0)}$

3) Consider the differential equation

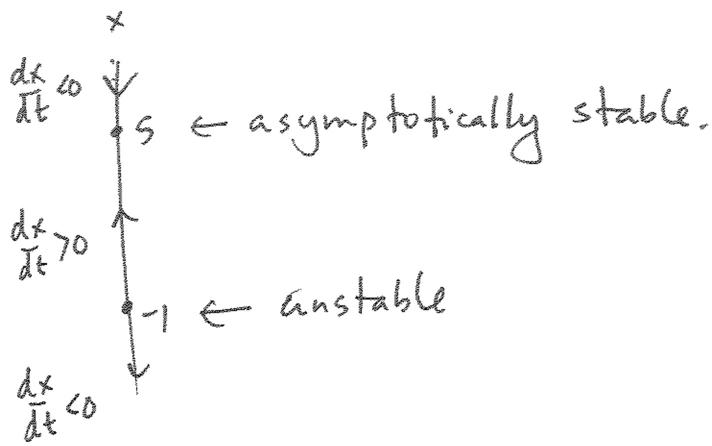
$$2 \cdot \frac{dx}{dt} = -x^2 + 4 \cdot x + 5 .$$

Find the equilibrium solutions for this differential equation and use a phase portrait to determine their stability. (If there was more time, I'd ask you to solve this differential equation.)

(10 points)

$$2 \frac{dx}{dt} = 0. = -(x^2 - 4x - 5)$$
$$= -(x-5)(x+1)$$

$x = 5, -1$ equil. sol'ns



4a) Consider the differential equation

$$\frac{d^2x}{dt^2} + 2\cdot\left(\frac{dx}{dt}\right) + 5\cdot x = 0,$$

which could arise as a model of a damped mass spring system. Find the general solution to this homogeneous differential equation.

(10 points)

$x = e^{rt}$

$p(r) = r^2 + 2r + 5 = (r+1)^2 + 4 = (r+1+2i)(r+1-2i)$

$r = -1 \pm 2i \qquad e^{(-1+2i)t} = e^{-t}(\cos 2t + i\sin 2t)$

$$\boxed{x_H(t) = e^{-t}\left(c_1\cos 2t + c_2 \sin 2t\right)}$$

4b) Now consider the non–homogenous differential equation

$$\frac{d^2x}{dt^2} + 2\cdot\left(\frac{dx}{dt}\right) + 5\cdot x = 10\cdot\cos(t).$$

(Such differential equations arise if, in addition to the drag and spring forces, the mass is subjected to a time–periodic sinusoidal external force.) Find all solutions to this non–homogeneous differential equation.

(10 points)

try

$5(\; x_p = A\cos t + B\sin t$

$2(\; x' = -A\sin t + B\cos t$

$+ 1(\; x'' = -A\cos t - B\sin t$

$L(x_p) = \cos t\,[5A + 2B - A] \overset{?}{=} 10\cos t$

$\quad + \sin t\,[5B - 2A - B].$

$4A + 2B = 10 \qquad 2A + B = 5$

$-2A + 4B = 0 \qquad -A + 2B = 0$

$$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$x_p = 2\cos t + \sin t$

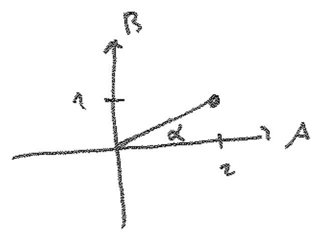
$x = x_p + x_H$

$= 2\cos t + \sin t$

$\quad + e^{-t}(c_1\cos 2t + c_2\sin 2t)$

4c) Notice that because of the damping in this problem, every solution to the non–homogeneous differential equation above actually converges to the particular solution you found as part of your work in (4b), as $t \rightarrow \infty$. Therefore this particular solution is called the "steady periodic" solution. (This is reminiscent of what happened in your Newton's Law of Cooling Maple project.) Put this steady periodic solution into amplitude–phase form.

(10 points)



$$x_{sp}(t) = \underset{A}{2}\cos t + \underset{B=1}{\sin t}$$
$$= C\cos(t-\alpha)$$

$$C = \sqrt{A^2+B^2} = \sqrt{5}$$

$$\frac{A}{C} = \cos\alpha \qquad\qquad \alpha = \tan^{-1}\left(\tfrac{1}{2}\right)$$

$$\frac{B}{C} = \sin\alpha$$

$$B/A = \tan\alpha$$

4d) Discuss how the amplitude and time delay of this steady periodic responce function compare to those of the external forcing function $10 \cdot \cos(t)$.

(5 points)

the amplitude $\sqrt{5}$ of the responce depends of the mass as well as the periodic force amplitude, ~~so~~ (as well as the damping.).

the time delay is just α, since $\omega = 1$.

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5) Let $L: V \to W$ be a linear transformation between vector spaces, and let $w \in W$. Suppose $L(v_P) = w$. Prove that $L(v) = w$ if and only if $v = v_P + v_H$, where v_H satisifes $L(v_H) = 0$. (This is the underlying justification for our approach to solving higher order linear differential equations.)

(5 points)

If $v = v_p + v_H$ then $L(v_p + v_H) = L(v_p) + L(v_H)$ L Linear

$= w + 0$

so v solves $L(v) = w$.

If (any) v solves $L(v) = w$ then

$v = v_p + (v - v_p)$

and $L(v - v_p) = Lv - Lv_p = w - w = 0$

so $v - v_p$ ~~solv~~ $:= v_H$ solves $L(v_H) = 0$.