Math 2280-1
Wed. 4/6
Finish Tuesday notes on classification of equilibria in
autonomous linear systems (Theorem A p. 2)
and autonomous non-linear systems (Theorem B, see below right).

Then begin
$\boxed{6.3}$ population models. (These notes may last thru Friday.)

• predator-prey:

$$PP \quad \begin{cases} x'(t) = ax - pxy = x(a - py) & \text{prey} \\ y'(t) = -by + qxy = y(-b + qx) & \text{predator} \end{cases}$$

$a, b, p, q > 0$

natural region of interest: $x \geq 0, y \geq 0$

equil sol'ns:

$x = 0 \Downarrow y = 0$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$y = a/p \Downarrow x = \frac{b}{q}$, $\begin{bmatrix} b/q \\ a/p \end{bmatrix}$ in 1st quad.
is this equil stable?

missing in yesterday's notes:

Theorem B General n, non-linear.
If the linearization for $\vec{x}' = F(\vec{x})$ @ equilibrium sol'n $\vec{x}^*$ is
$\vec{u}' = A\vec{u}$
and if all evals of A have strictly negative real part, then $\vec{x}^*$ is asymptotically stable.
If some λ_i has positive real part, then $\vec{x}^*$ is unstable for the non-linear DE. All other cases are borderline.
(proof requires analysis estimates)

linearize:

$$\begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} ax - pxy \\ -by + qxy \end{bmatrix}$$

$$\text{Jacobian matrix} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} a - py & -px \\ qy & -b + qx \end{bmatrix}$$

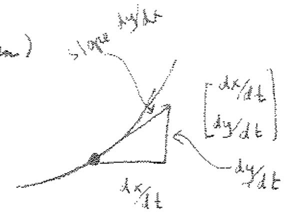
$$\text{at } \vec{x}_*, \quad J = \begin{bmatrix} 0 & -pb/q \\ aq/p & 0 \end{bmatrix}$$

$|J - \lambda I| = \lambda^2 + ab$; $\lambda = \pm i\sqrt{ab}$

linearization DE is stable center (ellipse trajectories)

this is borderline for the original non-linear system....

method for understanding sol'n trajectories: (in this case for the non-linear problem)



270 Chain rule $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

So $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ (use this in §9.1 #24 HW too!)

for PP, $\frac{dy}{dx} = \frac{y(-b+qx)}{x(a-py)}$ separable!

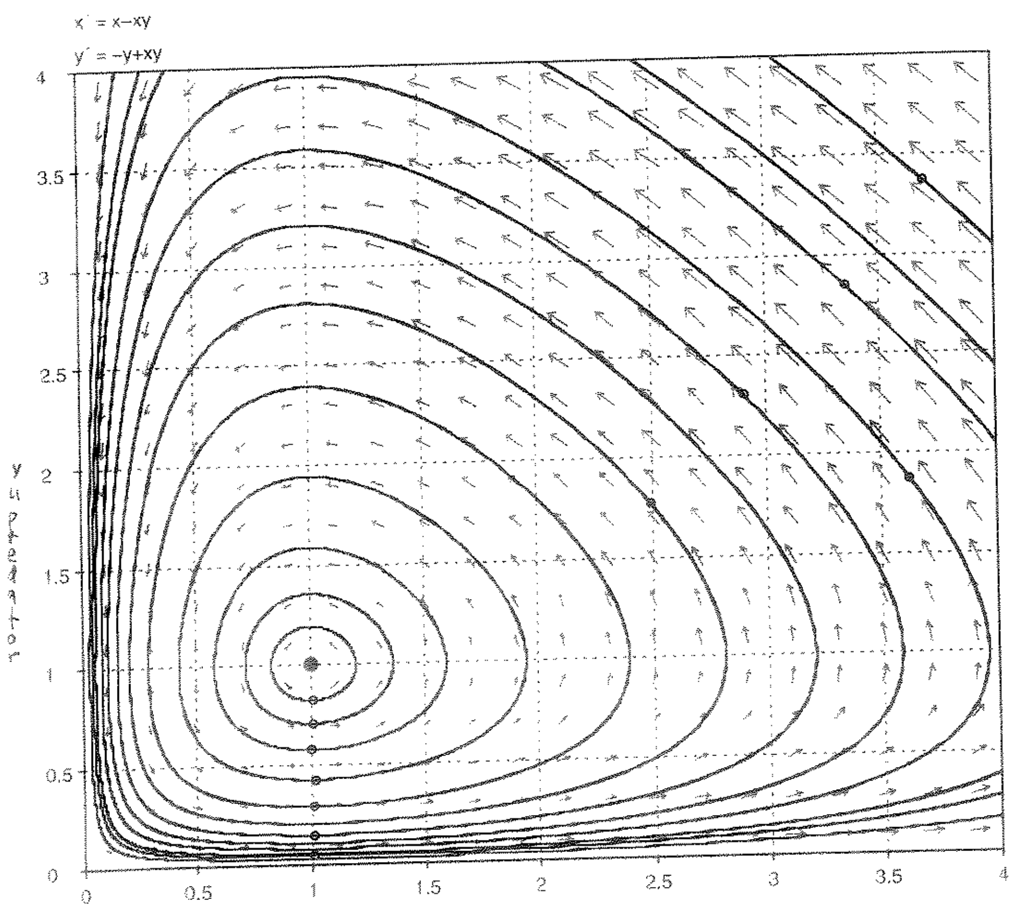
$\frac{a-py}{y}\,dy = \frac{-b+qx}{x}\,dx$

$a\ln y - py = -b\ln x + qx + C$ (in 1st quadrant!)

$\underbrace{a\ln y + b\ln x - py - qx}_{f(x,y)} = C$

So solution trajectories follow level curves of $f(x,y)$ (See figures 6.3.1 & 6.3.2 page 401)
and this $f(x,y)$ has a local max at $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b/q \\ a/p \end{bmatrix} = \vec{x}^*$ (Actually global 1st quadrant max!)

So $\vec{x}^*$ is stable, and general sol'ns are periodic!!



- Competition model

$$\frac{dx}{dt} = \underbrace{a_1 x - b_1 x^2}_{\text{logistic}} - \underbrace{c_1 xy}_{\text{competition}}$$

all params > 0

$$\frac{dy}{dt} = \underbrace{a_2 y - b_2 y^2}_{\text{logistic}} - \underbrace{c_2 xy}_{\text{competition?}}$$

Equil. sol'ns: $x\,[a_1 - b_1 x - c_1 y] = 0$
$y\,[a_2 - b_2 y - c_2 x] = 0$

$x = 0 \Rightarrow y(a_2 - b_2 y) = 0 \Rightarrow y = 0$ or $y = a_2/b_2$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ a_2/b_2 \end{bmatrix}$

$x \neq 0 \Rightarrow a_1 - b_1 x - c_1 y = 0 \Rightarrow y = 0$ or $a_2 - b_2 y - c_2 x = 0$

$y = 0 \Rightarrow x = \frac{a_1}{b_1}$

$\begin{bmatrix} a_1/b_1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} b_1 & c_1 \\ c_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

(book two page 540 (8))

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{b_1 b_2 - c_1 c_2}\begin{bmatrix} b_2 & -c_1 \\ -c_2 & b_1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

call this $\begin{bmatrix} x_E \\ y_E \end{bmatrix}$

$$= \frac{1}{b_1 b_2 - c_1 c_2}\begin{bmatrix} a_1 b_2 - c_1 b_1 \\ -a_1 c_2 + b_1 a_2 \end{bmatrix}$$

[this may or may not be in 1^{st} quadrant; depends on parameter values]

Text claims

1. If $c_1 c_2 < b_1 b_2$
(relatively low competition between species vs. self-inhibitions)
then $\begin{bmatrix} x_E \\ y_E \end{bmatrix}$ is asymptotically stable equil, and $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \to \begin{bmatrix} x_E \\ y_E \end{bmatrix}$

2. if $c_1 c_2 > b_1 b_2$, $\begin{bmatrix} x_E \\ y_E \end{bmatrix}$ is unstable, and
either $x(t)$ or $y(t) \to 0$ as $t \to \infty$
(with 100% probability)

(for any IVP if $x_0 > 0$, $y_0 > 0$, and if $\begin{bmatrix} x_E \\ y_E \end{bmatrix}$ is in 1^{st} quad.)

book shows examples - we'll work out the general proof!

Discussion of $\begin{bmatrix} x_E \\ y_E \end{bmatrix}$:

$$J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} a_1 - 2b_1 x - c_1 y & -c_1 x \\ -c_2 y & a_2 - 2b_2 y - c_2 x \end{bmatrix}$$

at $\begin{bmatrix} x_E \\ y_E \end{bmatrix}$ we have $\begin{bmatrix} b_1 & c_1 \\ c_2 & b_2 \end{bmatrix} \begin{bmatrix} x_E \\ y_E \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

so (trickery!) at this point

$$J = \begin{bmatrix} -b_1 x_E & -c_1 x_E \\ -c_2 y_E & -b_2 y_E \end{bmatrix}$$

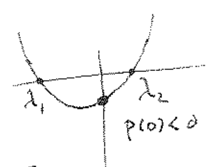
write $x = x_E$
$y = y_E$

$$p(\lambda) = |J - \lambda I| = (b_1 x + \lambda)(b_2 y + \lambda) - c_1 c_2 x y$$

$$p(\lambda) = \lambda^2 + (b_1 x + b_2 y)\lambda + (b_1 b_2 - c_1 c_2)\underbrace{xy}_{\substack{>0 \\ \text{by hypothesis}}}$$

the graph of $p(\lambda)$ is a concave up parabola

(2) if $c_1 c_2 > b_1 b_2$ then $p(0) < 0$
so $p(\lambda)$ has a positive and a negative root, by intermediate value thm, so $\begin{bmatrix} x_E \\ y_E \end{bmatrix}$ is (unstable) saddle.

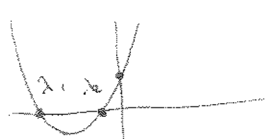


(1) if $c_1 c_2 < b_1 b_2$, then
vertex of parabola occurs at $\lambda = -\left(\frac{b_1 x + b_2 y}{2}\right)$ (complete square!)

and $p\left(-\frac{b_1 x + b_2 y}{2}\right) = \frac{1}{4}(b_1 x + b_2 y)^2 - \frac{1}{2}(b_1 x + b_2 y)^2 + (b_1 b_2 - c_1 c_2) x y$

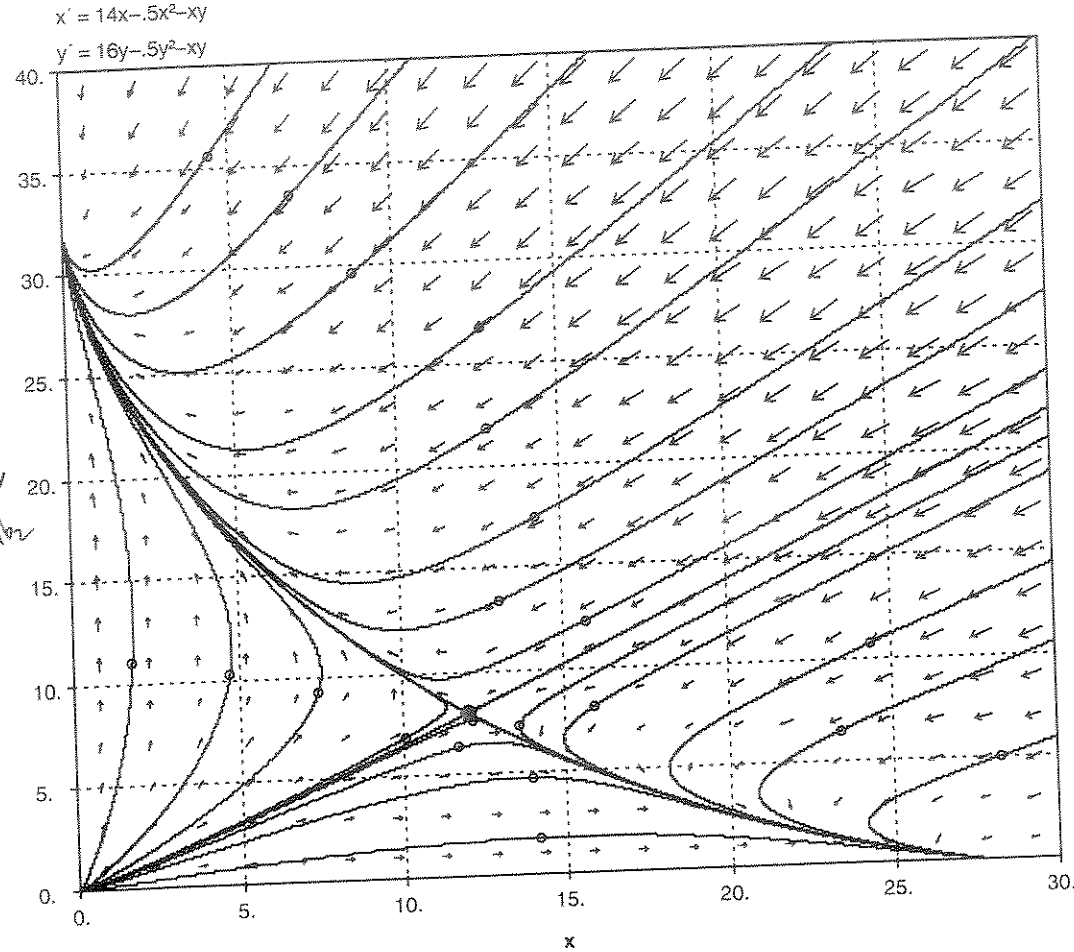
$$= -\frac{1}{4}\Big[\underbrace{b_1^2 x^2 + b_2^2 y^2 + 2b_1 b_2 x y \quad - 4b_1 b_2 x y}_{(b_1 x - b_2 y)^2} + 4c_1 c_2\Big]$$

$$< 0$$

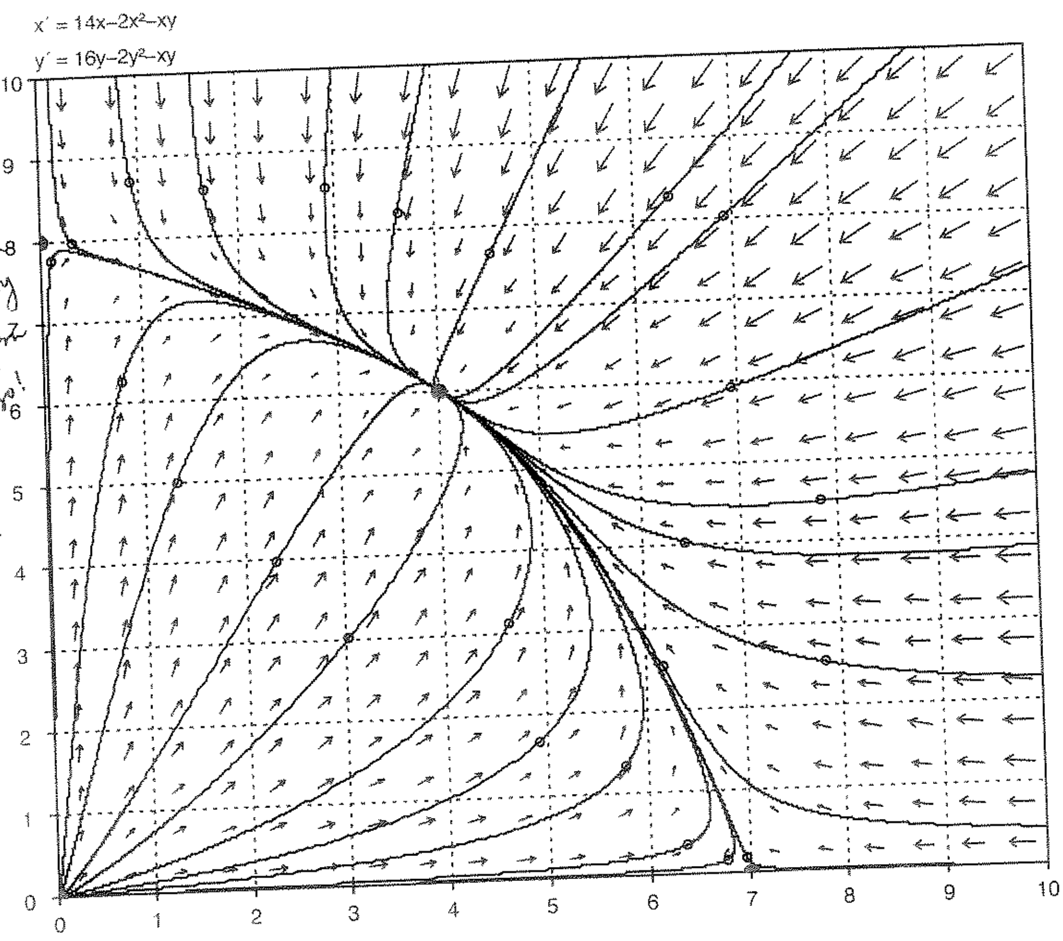


since $p(0) > 0$ in this case, $p(\lambda)$ has two negative roots. $\blacksquare$

(this proves the stability claims for the linearized problem, not the global
... (in the nonlinear problem)



$c_1c_2 > b_1b_2$

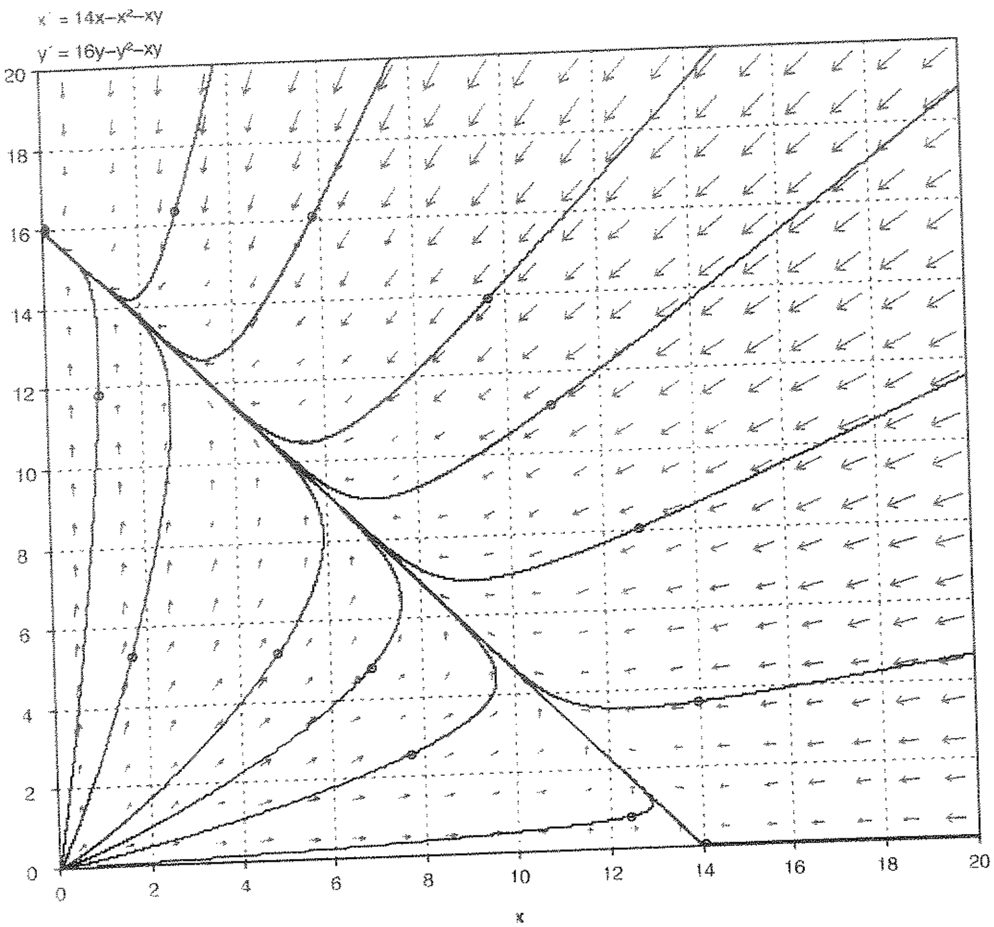


We studied exactly this system a few days ago!

$c_1c_2 < b_1b_2$

We didn't consider all possible cases, even of this "simple" competition model:

e.g.
$b_1 b_2 = c_1 c_2$



e.g.
$\begin{bmatrix} x_e \\ y_e \end{bmatrix}$ not in 1st quad.