Math 2280-1
Tuesday 4/5

- Finish example of competing species from Monday:

$$x' = 14x - 2x^2 - xy \quad \text{rabbits}$$
$$y' = \underbrace{16y - 2y^2}_{\text{logistic}} \underbrace{- xy}_{\text{competition}} \quad \text{squirrels}$$

we found 4 equilibrium soltns

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

- discuss linearization shortcut using Jacobian, page 5 Monday

then linearize this system via Jacobian, & compare to phase portrait, at the other three equilibria

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 7 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 8 \end{bmatrix}$

General picture:
(and nomenclature)

Eigenvalues of A	Type of Critical Point
Real, unequal, same sign	Improper node
Real, unequal, opposite sign	Saddle point
Real and equal	Proper or improper node
Complex conjugate	Spiral point
Pure imaginary	Center

FIGURE 6.2.9. Classification of the critical point (0, 0) of the two-dimensional system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Linear

Eigenvalues λ_1, λ_2 for the Linearized System	Type of Critical Point of the Almost Linear System
$\lambda_1 < \lambda_2 < 0$	Stable improper node
$\lambda_1 = \lambda_2 < 0$	Stable node or spiral point
$\lambda_1 < 0 < \lambda_2$	Unstable saddle point
$\lambda_1 = \lambda_2 > 0$	Unstable node or spiral point
$\lambda_1 > \lambda_2 > 0$	Unstable improper node
$\lambda_1, \lambda_2 = a \pm bi \quad (a < 0)$	Stable spiral point
$\lambda_1, \lambda_2 = a \pm bi \quad (a > 0)$	Unstable spiral point
$\lambda_1, \lambda_2 = \pm bi$	Stable or unstable, center or spiral point

FIGURE 6.2.12. Classification of critical points of an almost linear system.

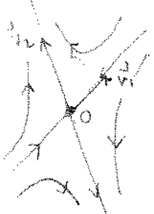
nonlinear - dominated by linearization (see Theorem B, page 4)

If λ real: $\vec{x}_H(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$

or (if λ_1 is defective)

$\vec{x}_H(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 (e^{\lambda_1 t})(t\vec{v}_1 + \vec{u})$

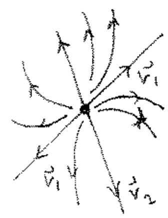
$\lambda_1 < 0 < \lambda_2$



saddle

(unstable)

$0 < \lambda_1 < \lambda_2$

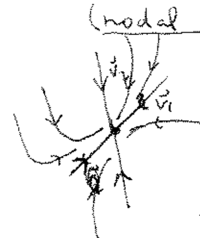


(unstable)
improper node
(or nodal source)

$\lambda_1 = \lambda_2$ "proper"

$\lambda_2 < \lambda_1 < 0$

stable improper node
(nodal sink)



$\lambda_1 = \lambda_2$ proper

If $\lambda = a + bi$ complex:

 $a < 0$ (asy. stable) spiral sink

 $a > 0$ (unstable) spiral source

 $a = 0$ (stable) center for linear; indeterminant for non-linear

Theorem A General n: Consider the DE $\vec{x}' = A_{n\times n}\vec{x}$. If the real part of each eigenvalue of A is negative then $x^* = \vec{0}$ is asymptotically stable equilibrium. If, on the other hand, any eigenvalue of A has positive real part then $\vec{x}^* = \vec{0}$ is unstable. If $Re(\lambda_i) \le 0 \ \forall i$, and some $Re\,\lambda_i = 0$ then $x^* = \vec{0}$ may or may not be stable, depending on whether or not the corresponding eigenspace(s) is defective.

proof: Can you fill this in using all of our work on linear systems of DE's??

$A_{2\times 2}$ complex eigenvalues yields spiral soltns (via e^{At} computations)

$A_{2\times 2}$ complex evals.

$$A(\vec{u}+i\vec{v}) = (a+bi)(\vec{u}+i\vec{v})$$

$$A\vec{u} + iA\vec{v} = a\vec{u} - b\vec{v} + i(b\vec{u} + a\vec{v})$$

$$A\vec{u} = a\vec{u} - b\vec{v}$$
$$A\vec{v} = b\vec{u} + a\vec{v}$$

$$A\underbrace{\left[\vec{u}\,\middle|\,\vec{v}\right]}_{S} = \left[\vec{u}\,\middle|\,\vec{v}\right]\underbrace{\begin{bmatrix} a & b \\ -b & a \end{bmatrix}}_{B}$$

$$AS = SB$$

$$A = SBS^{-1}$$

$$e^{At} = S e^{Bt} S^{-1}$$

soltn to

$$\begin{cases} \dfrac{d\vec{x}}{dt} = A\vec{x} \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

is $e^{At}\vec{x}_0$

$$= S\, e^{Bt}\, \underbrace{(S^{-1}\vec{x}_0)}_{\vec{z}_0}$$

$$= \underbrace{S\, \underbrace{e^{at} \underbrace{\begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}}_{\text{rotates } z_0 \ bt \text{ radians clockwise, so traces out a circle}}}_{\substack{a=0 \to \text{circle} \\ a<0 \to \text{spiral conv to } 0 \text{ as } t\to\infty \\ a>0 \to \text{spiral conv to } \infty \text{ as } t\to\infty}} \begin{bmatrix} \vec{z}_0 \end{bmatrix}}_{\substack{a=0 \to \text{ellipse} \\ a<0 \ \text{spiral sink} \\ a>0 \ \text{spiral source}}}$$

$$B = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = aI + bR.$$

$$e^{Bt} = e^{atI + btR}$$
$$= e^{atI}\, e^{btR} \qquad (I, R \text{ commute})$$
$$= e^{at} I \left(I + btR + \frac{(bt)^2}{2!}R^2 + \dots \right.$$

$$R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

$$R^3 = -R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$R^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

repeats.

$$\text{so } e^{Rt} = \begin{bmatrix} 1 - \frac{(bt)^2}{2!} + \frac{(bt)^4}{4!} - \dots & = - \\ -bt + \frac{(bt)^3}{3!} - \dots & = \end{bmatrix}$$

$$= \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}$$

$$= \mathrm{Rot}_{(-bt)} \qquad (\text{rotate } bt \text{ clockwise})$$

$$\text{so } e^{Bt} = e^{at}\begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}$$

Test your complex mettle by linearizing

$$x' = x - y - x^2 + xy$$
$$y' = -y - x^2$$

about the equilibrium $(-1,-1)$ (6.1.8)