Math 2280-1
Monday 4/4

HW for Friday 4/8 ①

6.1 5, 8, 11, 15, 20, 24 use pplane to visualize your work in this section, as the directions indicate. But no need to hand in hardcopies of pplane output from this section

6.2 5, 6, 7, 8, 9, 14, 15, 19, 27, 30 On 19, 27 use pplane to find and classify the other equil. sol'tns besides the origin, print out and hand in phase portrait justification (with sample sol'tn trajectories). (For fun you can linearize about these equil. and compare to non-linear pict)

## Chapter 6 introduction : non-linear systems of DE's.

e.g.

$$(1)\ \begin{cases} \frac{dx}{dt} = F(x,y,t) \\ \frac{dy}{dt} = G(x,y,t) \end{cases}$$

system of 2 $1^{st}$ order DE's.

example (6.3)

$x(t)$ = prey population (fish, rabbits, etc.)

$y(t)$ = predator population (sharks, foxes, etc.)

$$\begin{cases} \frac{dx}{dt} = ax - pxy \\ \frac{dy}{dt} = -by + qxy \end{cases}$$

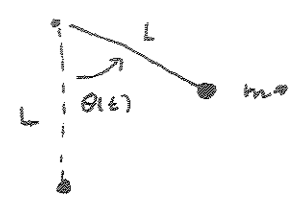
($-cx^2$ if you want logistic prey)

explain model assumptions:

example (6.4)

$x(t)$ = pendulum angle $\Theta(t)$

$y(t) = x'(t)$ = angular velocity



we've derived

$$x'' + \frac{g}{L}\sin x = 0, \text{ so}$$

$$\begin{cases} x' = y \\ y' = -\frac{g}{L}\sin x \end{cases}$$

Def: If the only dependence of F & G on t in (1) is through $x(t)$ & $y(t)$ (as in the 2 examples above), the system is called autonomous, i.e.

$$(2)\ \begin{cases} \frac{dx}{dt} = F(x,y) \\ \frac{dy}{dt} = G(x,y) \end{cases}$$

In this case we call the x-y plane the phase plane, and the solution curves $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ are called trajectories. They follow the tangent vector field $\begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$.

constant sol'ns to (2) are called <u>equilibrium solutions</u>
They are exactly the sol'ns to the (nonlinear) system

$$(3) \quad \begin{cases} 0 = F(x,y) \\ 0 = G(x,y) \end{cases}$$

<u>example</u> <u>competing</u> <u>species</u>, say $x(t)$ = rabbit population
$y(t)$ = squirrel population

perhaps

$$\frac{dx}{dt} = 14x - 2x^2 - xy$$

$$\frac{dy}{dt} = \underbrace{16y - 2y^2}_{\text{logistic}} - \underbrace{xy}_{\text{competition.}}$$

<u>Find the equilibrium sol'ns:</u>

ans $(0,0)$
$(0,8)$
$(7,0)$
$(4,6)$

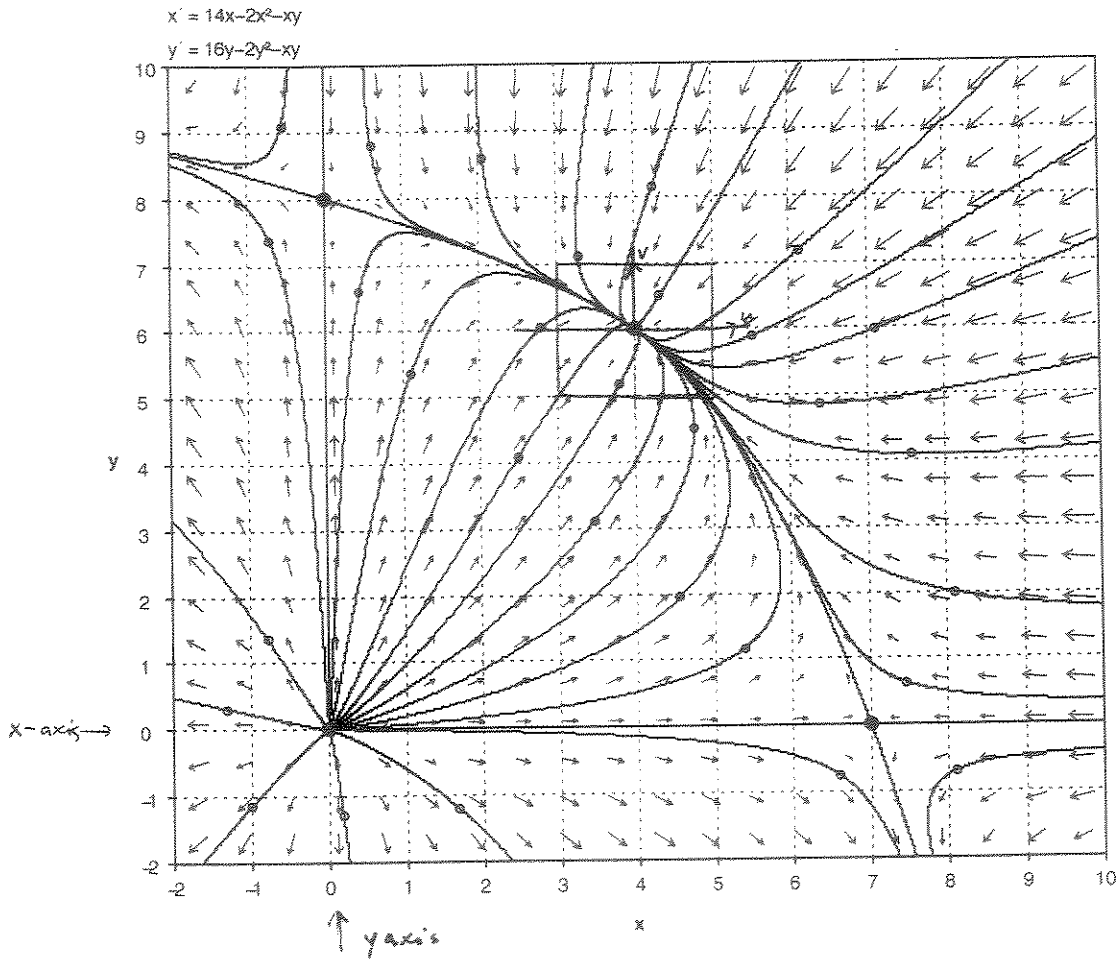
It will be important to know whether equilibrium sol'ns are <u>stable</u> or <u>unstable</u>:

<u>Def</u> $\begin{bmatrix} x_* \\ y_* \end{bmatrix} = \vec{x}^*$ is a <u>stable equilibrium</u> for (2) if it is a const (equil) soltion, i.e. satisfies (3),
<u>and</u> if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.
whenever $\|\vec{x}^* - \vec{x}_0\| < \delta$ $\quad \left( \|\vec{x}^* - \vec{x}_0\| = \sqrt{(x_0 - x_*)^2 + (y_0 - y_*)^2} \right)$
then the sol'n to (2) with $\vec{x}(0) = \vec{x}_0$
satisfies $\|\vec{x}(t) - \vec{x}^*\| < \varepsilon \quad \forall t > 0$.

$\begin{bmatrix} x_* \\ y_* \end{bmatrix}$ is <u>unstable equilibrium</u> if it is an equilibrium sol'n that is not stable.

$\begin{bmatrix} x_* \\ y_* \end{bmatrix}$ is <u>asymptotically stable</u> iff it's a stable equilibrium, <u>and</u> $\exists \delta > 0$
s.t. $\|\vec{x}^* - \vec{x}_0\| < \delta \Rightarrow$ sol'n to IVP with $\vec{x}(0) = \vec{x}_0$
satisfies $\lim_{t \to \infty} \vec{x}(t) = \vec{x}^*$

Here's the phase portrait (made with pplane) for our competition model:




What happens to rabbit-squirrel populations??

- notice that if we restrict this picture to either x-axis or y-axis (i.e. one pop $\equiv 0$) we get the Chapter 2 1-dim'l phase portraits for logistic DE's.

- discuss apparent stability of the 4 equilibria.

- discuss apparent long term population behavior, assuming both $x_0, y_0 > 0$.

<u>Linearization</u> : Near an equilibrium point, the non-linear system can be effectively approximated with a linear one:

<u>Example</u>: linearize rabbit-squirrel model near $\begin{bmatrix}4\\6\end{bmatrix}$:

let $x = 4 + u$ $\quad$ $u = u(t)$ small in abs. val.
$\phantom{let}$ $y = 6 + v$ $\quad$ $v = v(t)$ small " "

then
$$\frac{du}{dt} = \frac{dx}{dt} = 14x - 2x^2 - xy = 14(4+u) - 2(4+u)^2 - (4+u)(6+v)$$
$$= \cancel{56} + 14u \cancel{-32} - 16u - 2u^2 \cancel{-24} - 6u - 4v - uv$$
$$= -8u - 4v \quad -2u^2 - uv$$

$$\frac{dv}{dt} = \frac{dy}{dt} = 16y - 2y^2 - xy = 16(6+v) - 2(6+v)^2 - (4+u)(6+v)$$
$$= \underbrace{96-72-24}_{0} + u(-6) + v(16-24-4) + v^2(-2) + uv(-1)$$
$$= -6u - \overset{12}{\cancel{8}}v \quad -2v^2 - uv$$

$$\begin{bmatrix}\frac{du}{dt}\\ \frac{dv}{dt}\end{bmatrix} = \underbrace{\begin{bmatrix}-8 & -4\\ -6 & -12\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix}}_{\text{linear piece}} + \underbrace{\begin{bmatrix}-2u^2-uv\\ -2v^2-uv\end{bmatrix}}_{\text{error}}$$

linear piece

error: if $\left\|\begin{bmatrix}u\\v\end{bmatrix}\right\| < \delta$ is small, then $\|\text{error}\| \leq 3\sqrt{2}\,\delta^2$ is tiny.

in the linearization, discard the error term!

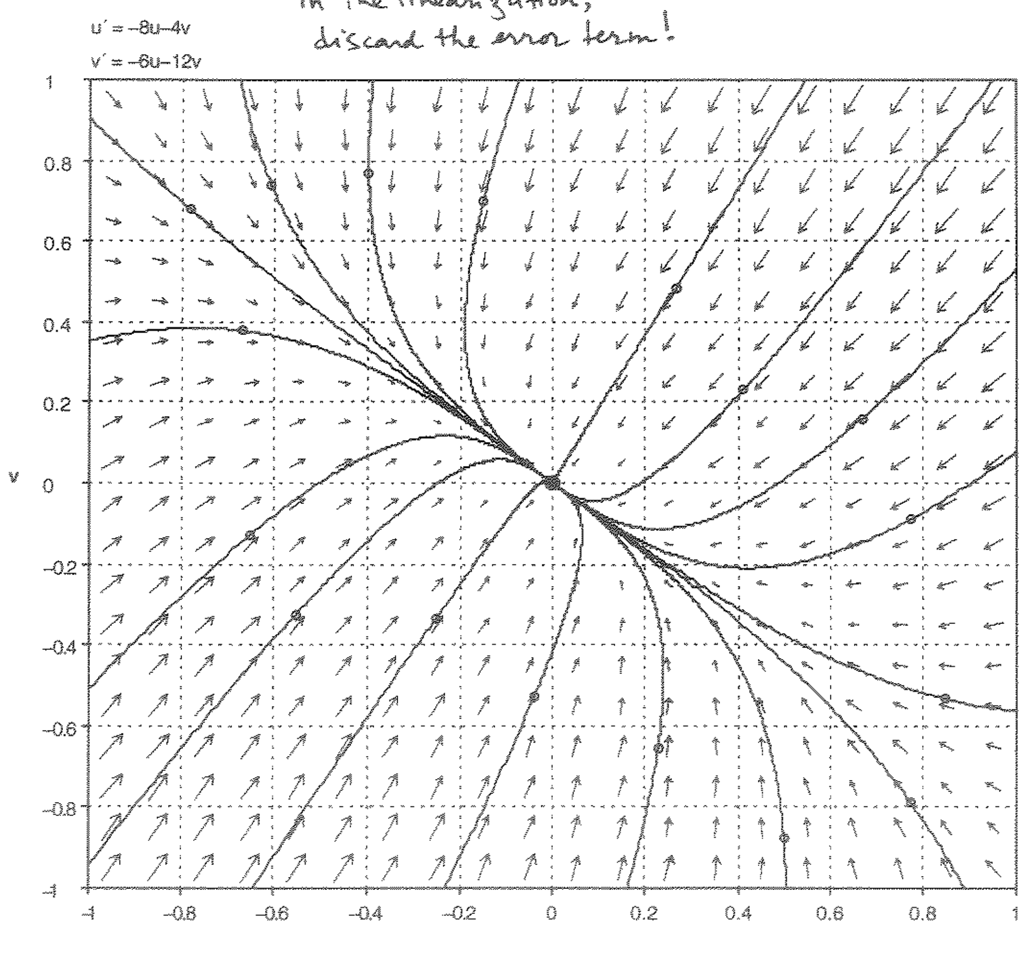
Matrix A has eigendata:

$\lambda = -4.7$ , $\lambda = -15.3$

$\vec{v}_1 = \begin{bmatrix}.77\\ -.64\end{bmatrix}$ $\qquad$ $\vec{v}_2 = \begin{bmatrix}.49\\ .89\end{bmatrix}$

$$\begin{bmatrix}u(t)\\ v(t)\end{bmatrix} \approx c_1 e^{-4.7t}\vec{v}_1 + c_2 e^{-15.3t}\vec{v}_2$$

notice how the translation change of variables and linearization yields a picture which essentially just magnifies the little box on page 3!



$\boxed{\text{Shortcut to linearization:}}$ (works for systems of $n$ DE's; illustrated for $n=2$)

Let
$$(1)\quad \begin{cases} x' = F(x,y) \\ y' = G(x,y) \end{cases}$$

$$F(x_*, y_*) = F(P) = 0$$
$$G(x_*, y_*) = G(P) = 0$$

write $\quad x(t) = x_* + u(t)$
$\quad\quad\quad\; y(t) = y_* + v(t)$

we are interested in what happens for $\|(u,v)\|$ small.

error; $\frac{\varepsilon}{\|(u,v)\|} \to 0$ as $(u,v) \to (0,0)$

$$x' = F(x_*+u, y_*+v) = \underbrace{F(x_*, y_*)}_{0} + F_x(x_*, y_*)u + F_y(x_*, y_*)v + \varepsilon_1(u,v)$$
$$y' = G(x_*+u, y_*+v) = \underbrace{G(x_*, y_*)}_{0} + G_x(-)u + G_y(-)v + \varepsilon_2(u,v)$$

Math 2210 affine approx.

$$u' = x' = F_x u + F_y v + \varepsilon_1(u,v)$$
$$v' = y' = G_x u + G_y v + \varepsilon_2(u,v)$$

where the partial derivs of $F$ & $G$ are evaluated at the equil. pt.

$$(2)\quad \begin{bmatrix} u' \\ v' \end{bmatrix} \cong \underbrace{\begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}}_{\text{"A"}} \begin{bmatrix} u \\ v \end{bmatrix}$$

(Partial derivs in $A$ are evaluated at $(x^*, y^*)$)

this is the linearization of (1), at $(x^*, y^*)$.
the eigenvector data of $A$ determines stability for the nonlinear system (1), in the non borderline cases.

the matrix $A$ is called the <u>Jacobian matrix</u> for $\vec{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$, at $\begin{bmatrix} x_* \\ y_* \end{bmatrix}$

(We use the same letters $u, v$ as we did for the non-linear problem, even though the sol'n $\begin{bmatrix} u \\ v \end{bmatrix}$ to the linearized problem only approximates the translated sol'n $\begin{bmatrix} u \\ v \end{bmatrix}$ to the non-linear problem)

<u>Exercise</u>
Check your page 4 work with the Jacobian shortcut!