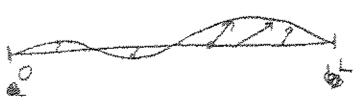
Math 2280-1
Tues 4/26

play day. We'll use Monday's and Today's notes to discuss the 1 space dimension heat & wave partial differential equations; how to build special building blocks which are products of sinusoidal functions in the space variable x, times functions of time; how to use superposition & Fourier series to solve the natural initial boundary value problems. We won't focus on the derivations of these PDE's, but they're included in the written notes.
Maple demos!

§9.6
Vibrating strings & the wave equation



vibrating string.

We assume tension in string changes as it is stretched,

i.e. $T = \mathbb{T}(\rho)$ $\quad \rho$ = density $\quad$ (probably $\mathbb{T}(\rho) \downarrow$ fcn ρ)

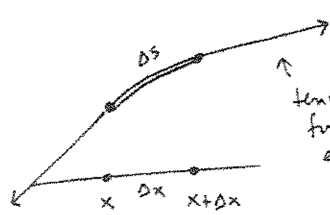
$T = \mathbb{T}(\rho_0) + \mathbb{T}'(\rho_0)(\rho - \rho_0)$ $\quad$ linearized model; $\quad T_0, \rho_0$ for stationary (but stretched) string.

$\mathbb{T}(\rho_0) = T_0$

<u>case 1</u> transverse (vertical) vibrations

$y = y(x,t)$ = vertical displacement

isolate a tiny piece, apply Newton's Law, linearize, to deduce wave eqtn:



tension force on ends

$$\underbrace{\rho\,\Delta s}_{\text{mass}}\ \underbrace{y_{tt}}_{\text{accel}} = \overbrace{\underset{\text{tension}}{T}\ \frac{y_x}{\sqrt{1+y_x^2}}\Bigg]_x^{x+\Delta x}}^{\text{net forces}}$$

vertical component of unit tang. vector to profile curve

$$\begin{bmatrix} 1 \\ y_x \end{bmatrix} \frac{1}{\sqrt{1+y_x^2}}$$

linearize

$$\frac{\Delta s}{\Delta x} \approx \sqrt{1+y_x^2} \approx 1$$

(y_x small $\Rightarrow y_x^2$ negligible)

$$\rho\,\Delta s = \rho_0\,\Delta x$$

$$\Rightarrow \rho = \rho_0 \frac{dx}{ds} = \rho_0 \sqrt{1+y_x^2} \approx \rho_0$$

$$\Rightarrow T \approx T_0$$

$$\rho_0 (\Delta x)\, y_{tt} \approx T_0 \left(y_x(x+\Delta x) - y_x(x) \right)$$

$$\rho_0\, y_{tt} \approx T_0 \left(\frac{y_x(x+\Delta x) - y_x(x)}{\Delta x} \right)$$

$$y_{tt} = \left(\frac{T_0}{\rho_0}\right) y_{xx}$$

$$\boxed{y_{tt} = a^2 y_{xx}}$$

$a = \sqrt{\dfrac{T_0}{\rho_0}}$ turns out to be wave speed.

<u>case 2</u> longitudinal motion, i.e. in direction of string, you get

$$x_{tt} = b^2 y_{xx}$$

where $b^2 = -T'(\rho_0)$, different speed!

What does "a" mean?

Special solution:

Let $f(z)$ a function of 1-variable.

consider $y(x,t) = f(x-at)$

$y_t = f'(x-at)(-a)$ chain rule!

$y_{tt} = f''(x-at)\,a^2$

$y_x = f'(x-at)\,1$

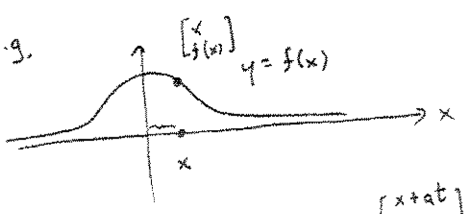
$y_{xx} = f''(x-at)\,1$

$y_{tt} = a^2 y_{xx}$!

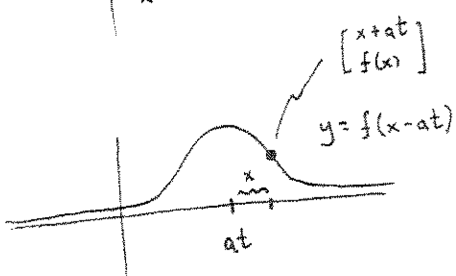
(also, $y(x,t) = f(x+at)$ solves the wave equation.)

$y(x,t) = f(x-at)$ is a wave (with constant "profile") moving to the right $\boxed{\text{with speed } a!}$

e.g.



profile at $t=0$



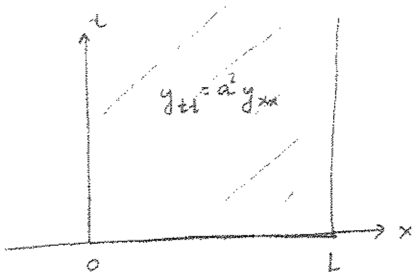
profile at time t has moved over by amount at

$y(x,t) = f(x+at)$ is a wave moving to the left!

(It will turn out that every solution to the wave eqtn $y_{tt} - a^2 y_{xx} = 0$ is a superposition of waves traveling to the left or right with speed a!)

In case 1, $a = \sqrt{\frac{T_0}{\rho_0}}$, so higher tension $\Rightarrow$ faster speed
lower density $\Rightarrow$ faster speed.

Natural IBVP's for wave eqtn:



$$\begin{cases} y_{tt} = a^2 y_{xx} & t>0,\ 0<x<L \\ y(x,0) = f(x) & \text{initial displacement} \\ y_t(x,0) = g(x) & \text{initial velocity} \end{cases}$$

plus (1) $y(0,t) = y(L,t) = 0$ fixed endpoints type 1

or (2) $y_x(0,t) = y_x(L,t) = 0$ free endpoints type 2

Use Fourier series & superposition: $(2 \times 2 = 4)$, and you can solve type 1 & type 2 problems!

(1) $\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{an\pi}{L}t\right)$

$y(x,0) = f(x)$ $\quad (y_t(x,0) = 0)$

$y(0,t) = y(L,t) \equiv 0$

$\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{an\pi}{L}t\right)$

$y_t(x,0) = g(x)$ $\quad (y(x,0) = 0)$

$y(0,t) = y(L,t) = 0$

(2) $\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{an\pi}{L}t\right)$

$y(x,0) = f(x)$ $\quad (y_t(x,0) = 0)$

$y_x(0,t) = y_x(L,t) = 0$

$\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{an\pi}{L}t\right)$

$y_t(x,0) = g(x)$ $\quad (y(x,0) = 0)$

$y_x(0,t) = y_x(L,t) = 0$