Math 2280-1
Monday 4/26

- p. 4-6 Friday notes; §9.4 forced oscillations revisited.

Remark: an equivalent way to solve

$L(x) = mx'' + cx' + kx = f(t)$, $f(t)$ 2L-periodic

$\rightarrow$ $f(t)$ has known Fourier series $f(t) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi}{L}t + \sum b_n \sin\frac{n\pi}{L}t$.

Write $x(t) = \frac{A_0}{2} + \sum A_n \cos\frac{n\pi}{L}t + \sum B_n \sin\frac{n\pi}{L}t$

compute $L(x)$, and equate Fourier coeff's to deduce $\frac{A_0}{2}, A_n, B_n$ (except when resonance)

9.5 1 space dimension heat equation for temperature $u(x,t)$ ← time $t \geq 0$

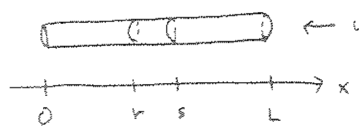
$\uparrow$ location $0 \leq x \leq L$

$$\boxed{u_t = k\,u_{xx}}$$

(or $u_t = k u_{xx} + f(x,t)$ for non-homogeneous heat eqtn)

"k" is called thermal diffusivity, depends on material, see table p. 622

derivation



← cross-sectional area a

"Heat" is a measurement of energy

If $u(x,t)$ is absolute temp (e.g. °Kelvin)

then the energy in test interval

$r \leq x \leq s$ is

$$E(t) = \int_r^s u(x,t)\, c\, \delta\, \underbrace{a\, dx}_{dV \ \text{cm}^3}$$

δ: density gm/cm³; $\underbrace{\delta\, a\, dx}_{dm \ \text{mass gm}}$

c: specific heat calories/gm

$\underbrace{= \text{energy required to heat 1 gm by } 1^\circ C}$

$\underbrace{c\,\delta\, a\, dx = \text{energy required to heat } dx\text{-piece } 1^\circ.}$

$u(x,t)\, c\,\delta\, a\, dx = $ energy in dx-piece.

Assuming lateral insulation

(so heat can only escape through ends),

assume

$$\frac{dE}{dt} = Ka\,(u_x(s) - u_x(r))$$

Heat flux proportional to temperature gradient

$$\int_r^s u_t\, c\delta a\, dx = Ka \int_r^s u_{xx}\, dx \quad (\text{FTC})$$

$\lim_{s\to r} \frac{1}{s-r}$: $\Rightarrow u_t\, c\delta a = Ka\, u_{xx}$

$$\boxed{u_t = k u_{xx}, \qquad k = \frac{K}{c\delta}}$$

- If u satisfies $u_t = k u_{xx}$ then so does $v = c_1 u + c_2$, so you may use Celsius, Farenheit, Kelvin

---

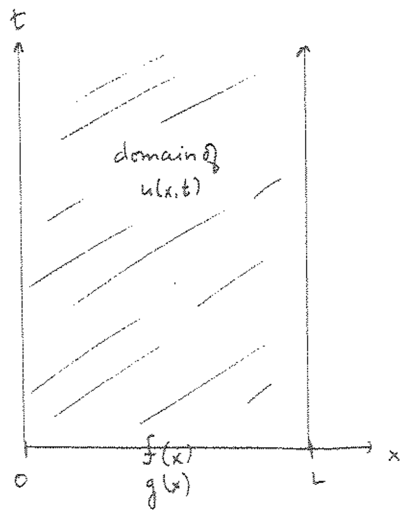
- $L(u) = u_t - k u_{xx}$ is linear so principle of superposition holds.

Two of the most-usually studied IBVP's (Initial boundary value problems)

type ①

$$\begin{cases} u_t = k u_{xx} & 0 < t < \infty,\ 0 < x < L \\ u(x,0) = f(x) & \text{init. temp. } 0 < x < L \\ u(0,t) = 0 & t > 0 \\ u(L,t) = 0 & t > 0 \end{cases}$$

boundary temp held const. (need not always be zero).



type ②

$$\begin{cases} u_t = k u_{xx} & 0 < t < \infty,\ 0 < x < L \\ u(x,0) = g(x) & 0 < x < L \\ u_x(0,t) = 0 & t > 0 \\ u_x(L,t) = 0 & t > 0 \end{cases}$$

no heat flux thru boundary (insulated end condition)

<u>Example</u> ① $\boxed{f(x) = \sin\left(\frac{\pi}{L} x\right)}$

product soln!

$u(x,t) = v(t) \sin(\omega x)$, $v(0) = 1$

$u_t = v'(t) \sin \omega x$

$u_{xx} = v(t)(-\omega^2) \sin \omega x$

$$\Leftrightarrow \begin{cases} v' = -k\omega^2 v \\ v(0) = 1 \end{cases} \Leftrightarrow v(t) = e^{-k\omega^2 t}$$

so $\boxed{u(x,t) = \sin\left(\frac{\pi}{L} x\right) e^{-k\left(\frac{\pi}{L}\right)^2 t}}$ solves ①! also $\boxed{u_n(x,t) = \sin\left(\frac{n\pi}{L} x\right) e^{-k\left(\frac{\pi}{nL}\right)^2 t}}$

general sol'n, use <u>sine series for f</u> & <u>superpose</u>!

$$f \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{L} n x\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{L} n x\right) e^{-k\left(\frac{\pi}{L} n\right)^2 t}$$

② $g(x) = \cos\left(\frac{\pi}{L} x\right)$

$\boxed{u(x,t) = \cos\left(\frac{\pi}{L} x\right) e^{-k\left(\frac{\pi}{L}\right)^2 t}}$! (see LHS)

general sol'n, use cosine series for $g(x)$ & superpose product solns!

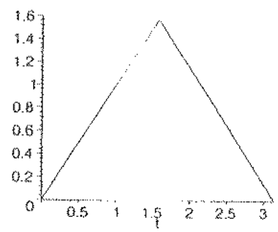
$$g \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{L} n x\right)$$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{L} n x\right) e^{-k\left(\frac{\pi}{L} n\right)^2 t}$$

(this method is called "separation of variables")

pictures next page, with tent function initial temperature, and $k = 1$

```
> tent:=t->(1-Heaviside(t-Pi/2))*t +Heaviside(t-Pi/2)*(Pi-t):
> plot(tent(t),t=0..Pi,color=black);
```



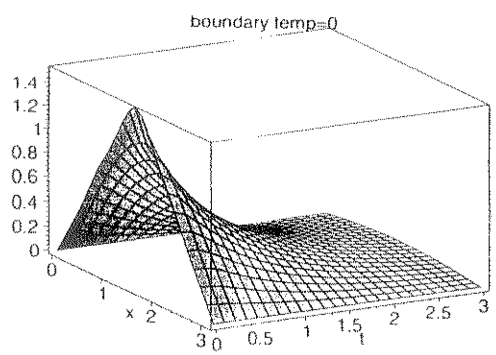

```
> sineseries:=proc(ff,L,b)   #ff=function, L=half period
  local m,  #dummy letter to index coefficients
        s;  #domain variable
  assume(m,integer):
  b:=m->simplify(2/L*int(ff(s)*sin(Pi/L*m*s),s=0..L)):
  end:
> sineseries(tent,Pi,B):
> B(n);
```

$$\frac{4\sin\left(\frac{\pi n\sim}{2}\right)}{\pi n\sim^2}$$

```
> u1:=(x,t)->sum(B(n)*sin(n*x)*exp(-n^2*t),n=1..10);
```

$$u1 := (x,t) \to \sum_{n=1}^{10} B(n)\sin(n\,x)\,e^{(-n^2 t)} \qquad (k=1)$$

```
> plot3d(u1(x,t),x=0..Pi,t=0..3,axes=boxed,title='boundary temp=0');
```



$$\begin{cases} u_t = u_{xx} \\ u(x,0) = tent(x) \\ u(0,t) = u(\pi,t) = 0 \quad t>0 \end{cases}$$

fixed boundary temp.

$\lim_{t\to\infty} u(x,t) = ?$

```
> cosseries:=proc(ff,L,a)   #ff=function, L=half period
  local m,  #dummy letter to index coefficients
        s;  #domain variable
  assume(m,integer):
  a:=m->simplify(2/L*int(ff(s)*cos(Pi/L*m*s),s=0..L)):
  end:
> cosseries(tent,Pi,A):
> A(0);
  A(n);
```

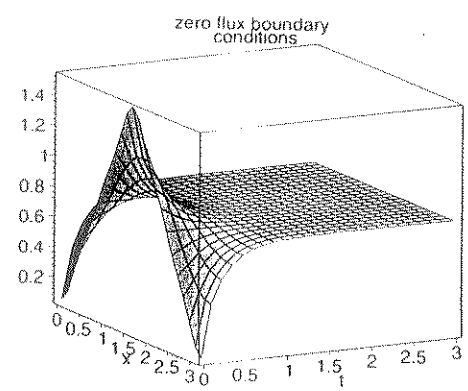
$$\frac{\pi}{2}$$

$$-\frac{2\left((-1)^{n\sim} - 2\cos\left(\frac{\pi n\sim}{2}\right) + 1\right)}{\pi n\sim^2}$$

```
> u2:=(x,t)->Pi/4+sum(A(n)*cos(n*x)*exp(-n^2*t),n=1..10);
```

$$u2 := (x,t) \to \frac{\pi}{4} + \left(\sum_{n=1}^{10} A(n)\cos(n\,x)\,e^{(-n^2 t)}\right) \qquad (k=1)$$

```
> plot3d(u2(x,t),x=0..Pi,t=0..3,axes=boxed,title='zero flux boundary
  conditions');
```



$$\begin{cases} u_t = u_{xx} \\ u(x,0) = tent(x) \\ u_x(0,t) = u_x(\pi,t) = 0 \end{cases}$$

no heat flux

$\lim_{t\to\infty} u(x,t) = ?$