Math 2280-1
Friday April 22

HW for Wed April 27
9.3 1, 9, 17, 20
9.4 1, 7, 9, 13 HW I ← today's notes
9.5 1, 2, 7
9.6 1, 5

§9.4 : forced oscillations via Fourier
(and 9.3 : even & odd extensions)

- At the end of Wednesday we checked how to derive Fourier series for piecewise continuous, 2L-periodic functions

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{L} n t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{L} n t\right)$$

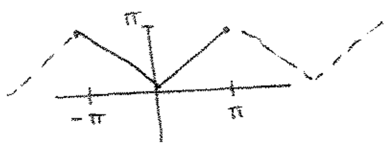
$$\langle f, g \rangle = \frac{1}{L}\int_{-L}^{L} f(t) g(t)\, dt$$

$$a_0 = \langle f, 1 \rangle = \frac{1}{L}\int_{-L}^{L} f(t)\, dt$$

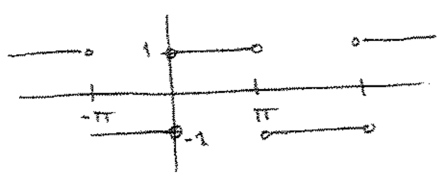
$$a_n = \langle f, \cos \tfrac{\pi}{L} n t \rangle = \frac{1}{L}\int_{-L}^{L} f(t) \cos\left(\frac{\pi}{L} n t\right) dt$$

$$b_n = \langle f, \sin \tfrac{\pi}{L} n t \rangle = \frac{1}{L}\int_{-L}^{L} f(t) \sin\left(\frac{\pi}{L} n t\right) dt$$

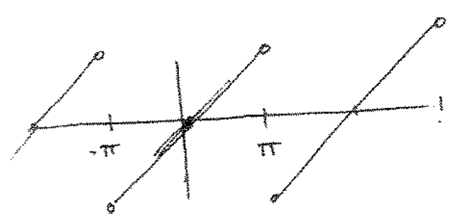
- We have computed Fourier series:



$$\text{tent}(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos nt}{n^2}$$

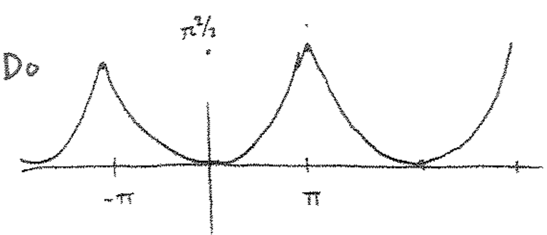


$$\text{square}(t) \sim \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin nt}{n} \qquad \left(= \frac{d}{dt}\,\text{tent}(t)\right)$$



$$\text{sawtooth}(t) \sim 2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nt}{n}$$

- Do



$$\text{parabola}(t) = \int_0^t \text{sawtooth}(\tau)\, d\tau \qquad \text{for } -\pi \le t \le \pi$$

$$\left(\frac{t^2}{2}\right) \qquad = 2\sum_{n=1}^{\infty} (-1)^n \frac{\cos nt}{n^2} + \frac{\pi^2}{6}$$

(See Wed notes).

- To get Fourier series linear combinations of fcns, take linear combos of Fourier series because Fourier coeff's are linear wrt the function input.

- We've notice that

  odd 2L-periodic functions have Fourier series with all $a_n = 0$,
  i.e. they are sine series

  even 2L-periodic functions have Fourier series with all $b_n = 0$
  i.e. they are cosine series

So, if $f : [0, L] \to \mathbb{R}$ is p.w. continuous
you can extend to $f : [-L, L] \to \mathbb{R}$
as even: $f(-t) := f(t) \quad 0 \le t \le L$
and then to $f : \mathbb{R} \to \mathbb{R}$ as 2L-periodic $\left(f(t + 2Ln) := f(t) \ \forall n \in \mathbb{Z} \right)$

$$\Rightarrow f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{L} n t\right) \qquad \text{cosine series for } f$$

OR
you can extend
to $f : [-L, L] \to \mathbb{R}$
as odd: $f(-t) := -f(t) \quad 0 \le t \le L$
and then to $f : \mathbb{R} \to \mathbb{R}$ as 2L-periodic

$$\Rightarrow f \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{L} n t\right) \qquad \text{sine series for } f$$

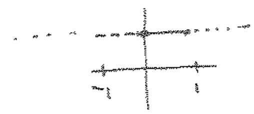
(useful in 9.5-9.6)

(also page 2.) Wed.

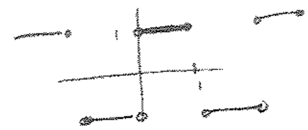
examples (we can steal)

1. Let $f(t) = 1$, $0 \le t \le 1$

   What is the cosine series for f?



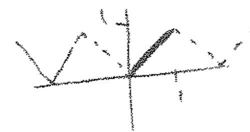
   What is the sine series for f?



Hint: Rescale square(t) from page 1

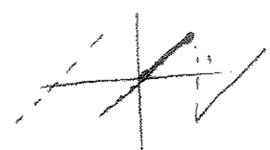
2. Let $g(t) = t$, $0 \le t \le 1$

   What is the cosine series for g?



Hint: page 1.

   What is the sine series for g?



Fourier & Springs §9.4

Consider the forced oscillation problem

$$mx'' + cx' + kx = f(t)$$

$$x'' + \frac{c}{m}x' + \frac{k}{m}x = \frac{f(t)}{m}.$$

It's equivalent to the first order system

$$x' = y$$

$$y' = -\frac{k}{m}x - \frac{c}{m}y + \frac{f(t)}{m}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & c/m \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{f(t)}{m} \end{bmatrix}$$

for which we have variation of parameters soln:

$$\vec{x}' = A\vec{x} + f$$

$$\vec{x}' - A\vec{x} = \vec{f}(t)$$

$$\vdots$$

$$\boxed{\vec{x}(t) = e^{At}\vec{x}_0 + e^{At}\int_0^t e^{-As}\vec{f}(s)\,ds.}$$

HW I Show that applying this general technique to

$$\begin{cases} x'' + x = f(t) \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

yields $\boxed{x(t) = \int_0^t f(s)\sin(t-s)\,ds + x_0\cos t + v_0\sin t}$

This formula is useful, but if we restrict to periodic forcing functions it's hard to decipher how $x(t)$ behaves. For example does forcing by f cause resonance?

Fourier series is another way to understand periodically forced mechanical systems, and will give precise answers about which $f(t)$ can cause resonance when there is no damping.

Consider the problem from page 4

$$\begin{cases} x'' + x = f(t) \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

and use the following table of particular soltns to deduce resonance/non resonance:

$x'' + \omega_0^2 x = 1 \qquad x_p = \frac{1}{\omega_0^2}$

$x'' + \omega_0^2 x = \cos\omega t \quad \begin{cases} x_p = \frac{1}{\omega_0^2-\omega^2}\cos\omega t & \omega \neq \omega_0 \\ x_p = \frac{1}{2}\frac{t}{\omega_0}\sin\omega_0 t & \omega = \omega_0 \end{cases}$

$x'' + \omega_0^2 x = \sin\omega t \quad \begin{cases} x_p = \frac{1}{\omega_0^2-\omega^2}\sin\omega t & \omega \neq \omega_0 \\ x_p = -\frac{1}{2}\frac{t}{\omega_0}\cos\omega_0 t & \omega = \omega_0 \end{cases}$

<u>Exercise 1</u> a) $x''(t) + x(t) = 3\cos t \qquad \omega_0 = 1 = \omega \qquad T_0 = 2\pi$

b) $x'' + x(t) = \underset{T=\pi}{\cos 2t} - \underset{T=2\pi/3}{2\sin 3t}$, so $f(t)$ has period $2\pi = T_0$

c) $x'' + x = \underset{T=2\pi}{3\cos t} + \underset{T=4\pi}{\sin t/2} \qquad f(t)$ has period $4\pi = 2T_0$.

<u>answer in general</u> let $f(t)$ have period $2L$

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi}{L}\right) t \qquad \text{Fourier series}$$

Consider the forced oscillation problem

$$x'' + \omega_0^2 x(t) = f(t)$$

- if no $\frac{n\pi}{L} = \omega_0$, then using page 5 particular soltns for $\omega = \frac{n\pi}{L}$ & infinite superposition

you get a bounded particular soltn

$$x(t) = \frac{a_0}{2\omega_0^2} + \sum_{n=1}^{\infty} \frac{a_n}{\omega_0^2 - \left(\frac{n\pi}{L}\right)^2} \cos\left(\frac{\pi}{L} n t\right) + \sum_{n=1}^{\infty} \frac{b_n}{\left(\omega_0^2 - \frac{n\pi}{L}\right)^2} \sin\left(\frac{\pi}{L} n t\right)$$

(since all a_n, b_n are uniformly bounded, series converges, uniformly in t, by ratio comparison test to $\sum \frac{1}{n^2}$)

- if some $\frac{N\pi}{L} = \omega_0$, <u>and</u> either a_N or $b_N \neq 0$ the corresponding x_p for this N will cause resonance.

$\updownarrow$

i.e. $L = \frac{N\pi}{\omega_0} = \frac{N\pi}{2\pi/T_0} = \frac{N}{2} T_0$; $\quad 2L = \text{forcing period} = N \cdot T_0$.

$\boxed{\text{So forcing period is a multiple of the natural period}}$

see page 5 examples.
see Maple!