Math 2280-1

Wednesday April 20

§9.3–9.4

Finish Tuesday notes:

- f' Fourier series by differentiating f's
- If f is a finite linear combo of 1, $\cos nt$'s, & $\sin nt$'s then that's its Fourier series

} pages 1–2

- 2L periodic fcn Fourier series

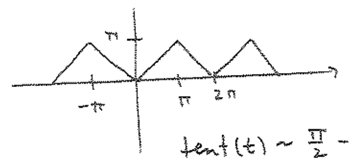
} page 3

§9.3

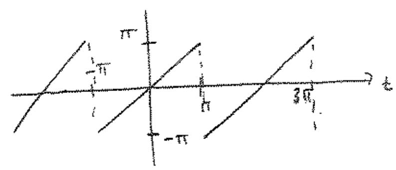
We've noticed that

<u>odd</u> 2L-periodic fcns' Fourier series are <u>sine series</u>, because all $a_n = 0$.

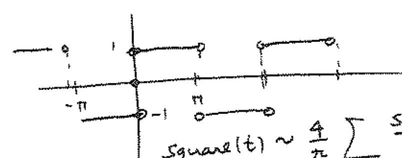
<u>even</u> 2L-periodic fcns' Fourier series are <u>cosine series</u> because all $b_n = 0$.



$$\text{tent}(t) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos nt}{n^2}$$



$$\text{sawtooth}(t) \sim 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nt}{n}$$



$$\text{square}(t) \sim \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin nt}{n}$$

So, If $f : [0, L] \to \mathbb{R}$

- you can extend $f : [-L, L] \to \mathbb{R}$ as an even fcn $f(t) := f(-t)$, and then to $\mathbb{R}$ as a 2L-periodic fcn $\left(f(t + 2kL) := f(t)\right)$ $-L < t < L$

$$\Rightarrow f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{L} n t\right)$$

<u>cosine series for f</u>

Or,

- you can extend $f : [-L, L] \to \mathbb{R}$ as an odd fcn $f(-t) := f(t)$ $0 < t < L$, and then to $\mathbb{R}$ as a 2L-periodic fcn

$$\Rightarrow f \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{L} n t\right)$$

<u>sine series for f</u>.

(on the interval $0 < t < L$ either series will converge to $f(t)$, wherever f is differentiable!)

(Useful in §9.5–9.6)

If f is 2L-periodic

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi}{L} n t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi}{L} n t\right)$$

$$\langle f, g \rangle = \frac{1}{L} \int_{-L}^{L} f(t) g(t)\, dt$$

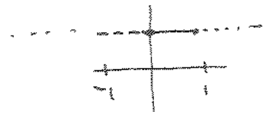
$$a_0 = \langle f, 1 \rangle = \frac{1}{L} \int_{-L}^{L} f(t)\, dt$$

$$a_n = \langle f, \cos \tfrac{\pi}{L} n t \rangle = \frac{1}{L} \int_{-L}^{L} f(t) \cos\left(\frac{\pi}{L} n t\right) dt$$

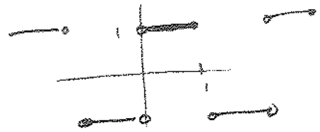
$$b_n = \langle f, \sin \tfrac{\pi}{L} n t \rangle = \frac{1}{L} \int_{-L}^{L} f(t) \sin\left(\frac{\pi}{L} n t\right) dt$$

examples (we can steal)

1. let $f(t) = 1, \quad 0 \le t \le 1$

   What is the cosine series for f?

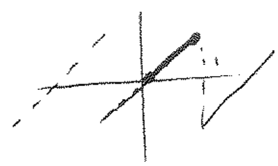
   

   What is the sine series for f?

2. let $g(t) = t, \quad 0 \le t \le 1$

   What is the cosine series for g?

   What is the sine series for g?

We talked about differentiating Fourier series term by term.
There is also:

<u>Theorem</u>: If f is p.w. cont., 2L-periodic, with Fourier series (page 605)

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} t\right) + b_n \sin\left(\frac{n\pi}{L} t\right)$$

integrated term by term.

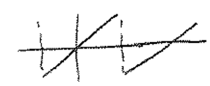
then the antiderivative

$$\int_0^t f(s)\,ds = \frac{a_0 t}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{L}{n\pi}\right) \sin\left(\frac{n\pi}{L} t\right) + b_n \left(\frac{L}{n\pi}\right)\left[-\cos\left(\frac{n\pi}{L} t\right) + 1\right]$$

series on the
right converges to this antideriv!

<u>Example</u> (9.3 #19)

start with $t = \text{sawtooth}(t) \sim 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nt}{n}$ $\quad -\pi < t < \pi$



Integrate once to deduce

$$\frac{t^2}{2} = 2 \sum_{n=1}^{\infty} (-1)^n \frac{\cos nt}{n^2} + \pi^2/6$$

(In HW need to get
all the way to $\frac{t^4}{24}$!)

Hint: use the average
value of $t^2/2$ on $[-\pi, \pi]$
to get the integration constant.