Math 2280-1
Tuesday April 19

Mostly we'll finish Monday's notes.
Then, continue:

if you want a written version of orthonormal bases, projection, and inner product spaces (as we discussed yesterday), I've posted links to 2270 notes from fall 2009.

Remark: We can actually prove the differentiation theorem: 3

if $f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$

and if

$$f' \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nt + \sum_{n=1}^{\infty} B_n \sin nt$$

"like" Laplace: Taking derivatives of $f(t)$ multiplies (and interchanges) Fourier coeffs by $\pm n$

$f: \{a_0, a_1, b_1, a_2, b_2, \ldots a_n, b_n \ldots\}$
$\downarrow$
$f': \{0, b_1, -a_1, 2b_2, -2a_2, \ldots nb_n, -na_n, \ldots\}$.

then

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(t)\,dt = \frac{1}{\pi}\left[f(\pi) - f(-\pi)\right] = 0 \quad \text{(since } f \text{ cont. \& } 2\pi\text{-peri)}$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(t)\cos nt\,dt = \frac{1}{\pi}\left[ f(t)\cos nt \Big|_{-\pi}^{\pi} - -n\int_{-\pi}^{\pi} f(t)\sin nt\,dt\right] = nb_n$$

$du = f'(t)dt \quad v = \cos nt$
$u = f(t) \quad dv = -n\sin nt\,dt$

and by same reasoning, $B_n = -na_n$ ■

Remark Because $\{\frac{1}{\sqrt{2}}, \cos t, \cos 2t, \ldots \cos nt, \sin t, \sin 2t, \ldots \sin nt\}$ are an orthonormal basis for their span $V_n$,

if $f(t) = \alpha_0 + \sum_{k=1}^{n} \alpha_k \cos kt + \sum_{k=1}^{n} \beta_k \sin kt \in V_n$

then it must be that

$\alpha_0 = \frac{a_0}{2}$
$\alpha_k = a_k$
$\beta_k = b_k$

the Fourier coeffs.

pf: e.g. $a_\ell = \langle f, \cos \ell t\rangle = \langle \alpha_0 + \sum_{k=1}^{n}\alpha_k \cos kt + \sum \beta_k \sin kt, \cos \ell t\rangle$
$= 0 + \alpha_\ell + 0$

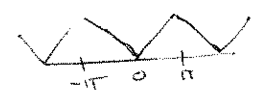
So if $f(t)$ already looks like a truncated Fourier series, that ~~is~~ is its Fourier series (so it would not be efficient to recompute Fourier coefficients).

<u>Example 4</u> What is the $2\pi$-periodic Fourier series for

a) $\sin 5t - 8\cos 10t$ ?

b) $\cos^2 3t$ ?

<u>Example 5</u> (magic formulas - more in Hw). We showed for "tent fcn" $f(t) = |t| \quad -\pi \le t \le \pi$




$$f(t) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos nt}{n^2}$$

by Conv. thm $f(t)$ = its Fourier series

a) Compute $f(\pi)$ Via Fourier to deduce $\displaystyle\sum_{n \text{ odd}} \frac{1}{n^2} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \dots = \frac{\pi^2}{8}$

b) Use (a) and cleverness to deduce $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots = \frac{\pi^2}{6}$

So far, we've assumed $f(t)$ is $2\pi$-periodic.

What if $g(u)$ is $2L$-periodic?

then the corresponding inner product $\langle g, h\rangle := \frac{1}{L}\int_{-L}^{L} g(u)h(u)\,du$

makes $\left\{ \frac{1}{\sqrt{2}},\ \cos\frac{\pi}{L}u,\ \cos\frac{2\pi}{L}u, \dots \cos\frac{k\pi}{L}u,\ \sin\frac{\pi}{L}u,\ \sin\frac{2\pi}{L}u, \dots \sin\frac{k\pi}{L}u, \dots \right\}$

orthonormal!

(proof: substitute $\frac{\pi}{L}u = t$; $u = \frac{L}{\pi}t$ to reduce to previous computations)

so the Fourier series for $g$ is defined by

$$g \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}u\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}u\right)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} g(u)\,du$$

$$a_n = \frac{1}{L}\int_{-L}^{L} g(u)\cos(nu)\,du$$

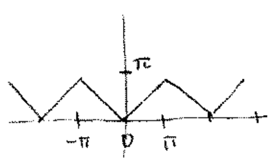
$$b_n = \frac{1}{L}\int_{-L}^{L} g(u)\sin(nu)\,du$$

(and then we use "t" instead of "u").

You can also just rescale Fourier series to get $2L$-periodic ones from $2\pi$-periodic ones

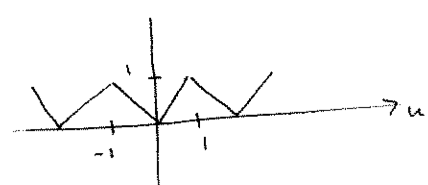
Example 6

did




$$f \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\cos nt}{n^2}$$

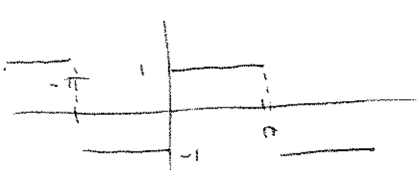
so,




$g(u) = \frac{1}{\pi}f(\pi u)$ so

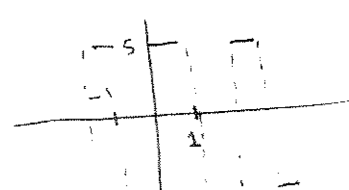
$g \sim$ ?

did




$$f \sim \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin nt}{n}$$

So




$g(u) = 5f(\pi u)$

so $g \sim$