Math 2280-1
Friday 4/15

Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 10.

Function	Transform	Function	Transform
✓ $f(t)$	$F(s)$	✓ e^{at}	$\frac{1}{s-a}$
✓ $af(t)+bg(t)$	$aF(s)+bG(s)$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
✓ $f'(t)$	$sF(s)-f(0)$	✓ $\cos kt$	$\frac{s}{s^2+k^2}$
✓ $f''(t)$	$s^2F(s)-sf(0)-f'(0)$	✓ $\sin kt$	$\frac{k}{s^2+k^2}$
1a $f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	$\cosh kt$	$\frac{s}{s^2-k^2}$
1b $\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$	$\sinh kt$	$\frac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$ 3a	✓ $e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$ 3b	✓ $e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$ (beautiful but skip)	$\frac{1}{2k^3}(\sin kt - kt\cos kt)$	$\frac{1}{(s^2+k^2)^2}$
2a $tf(t)$	$-F'(s)$ 2b	$\frac{t}{2k}\sin kt$	$\frac{s}{(s^2+k^2)^2}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$\frac{1}{2k}(\sin kt + kt\cos kt)$	$\frac{s^2}{(s^2+k^2)^2}$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\frac{e^{-as}}{s}$
$f(t)$, period p	$\frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\frac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\frac{1}{s}\tanh\frac{as}{2}$
t	$\frac{1}{s^2}$ 2c	$\left[\!\left[\frac{t}{a}\right]\!\right]$ (staircase)	$\frac{e^{-as}}{s(1-e^{-as})}$
t^n	$\frac{n!}{s^{n+1}}$		
$\frac{1}{\sqrt{\pi t}}$	$\frac{1}{\sqrt{s}}$		
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$		

HW due 4/22
7.3 3 7, 8, 17, 20, 31
7.4 2, 3, 36 (optional)
9.1 6, 7, 10 13 15, 17, 20, 30
9.2 2, 9

$$\mathcal{L}\{f(t)\}(s) := \int_0^\infty f(t)\, e^{-st}\, dt \qquad \text{for } |f(t)| \le Ce^{M},\ s > M.$$

$$\underset{F(s)}{\parallel}$$

1a) Use $\mathcal{L}\{f'(t)\}(s) = sF(s) - f(0)$ (Wed., integration by parts)
and mathematical induction
to show $\mathcal{L}\{f^{(n)}(t)\}(s) = s^n F(s) - s^{n-1} f(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0)$.

1b) Let $g(t) := \int_0^t f(\tau)\, d\tau$

notice $g'(t) = f(t)$
$g(0) = 0$

to show $G(s) = \dfrac{F(s)}{s}$

<u>Example</u>: did $\mathcal{L}\{1\}(s) = \frac{1}{s}$

find $\mathcal{L}\{t\}(s)$

$\mathcal{L}\{t^2\}(s)$

2a) Use limit def. of derivative to compute
$$F'(s).$$
Show it equals $\mathcal{L}\{-tf(t)\}(s)$.

2b) : Special cases
$$\mathcal{L}\{t\cos kt\}(s) = \frac{s^2-k^2}{(s^2+k^2)^2}$$

$$\mathcal{L}\{t\sin kt\}(s) = \frac{2ks}{(s^2+k^2)^2}$$

combine to deduce
$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{2k^2}\left(\frac{s^2+k^2}{(s^2+k^2)^2} - \frac{s^2-k^2}{(s^2+k^2)^2}\right)\right\}(t)$$
$$= \frac{1}{2k^3}\sin kt - \frac{1}{2k^2}t\cos kt$$

2c) $\mathcal{L}\{t^n\}(s) = \mathcal{L}\{t\cdot t^{n-1}\}(s) = \frac{n!}{s^{n+1}}$ by induction

<u>Example</u> Resonance revisited!

Solve the IVP

$$\begin{cases} x'' + \omega_0^2 x = F_0 \sin\omega t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

$\omega = \omega_0$

$\omega \neq \omega_0$

3a) Use def of $\mathcal{L}$ to show

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

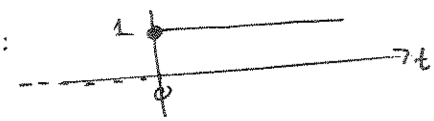
special cases: $\mathcal{L}\{e^{at}\cos kt\}(s) = \dfrac{s-a}{(s-a)^2+k^2}$ (done Wed, differently)

$\mathcal{L}\{e^{at}\sin kt\}(s) = \dfrac{k}{(s-a)^2+k^2}$

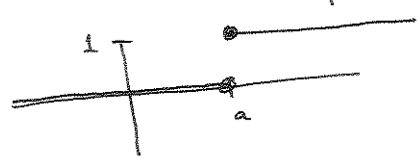
$\mathcal{L}\{e^{at}t^n\}(s) = \dfrac{n!}{(s-a)^{n+1}}$

3b) turning functions on and off.

$u(t) := \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$ graph of $u(t)$:



So $u(t-a) = \begin{cases} 1 & t \geq a \\ 0 & t < a \end{cases}$



"unit step function"
("Heaviside function" in Maple).

use def to show

$$\mathcal{L}\{u(t-a)f(t-a)\}(s) = e^{-as}F(s)$$

interesting Laplace patterns:

derivatives in t-space $\rightsquigarrow$ mult by s in transform space
mult by t is t-space $\rightsquigarrow$ derivs in s-space
mult by e^{at} in t-space $\rightsquigarrow$ translation by a is s-space
translation by a in t-space $\rightsquigarrow$ mult by e^{-as}

Example (p 484)

$m = 32$ lb ($m = 1$ slug), $c = 0$,

$k = 4$ lb/ft. mass initially at equilibrium ($x(0)=0, x'(0)=0$)

at $t=0$ apply $F(t) = \cos 2t$

at $t = 2\pi$ force is turned off. Solve IVP $\begin{cases} x'' + 4x = F(t) \\ x(0)=0 \\ x'(0)=0 \end{cases}$

$\downarrow x(t)$

soln

$s^2 X(s) - 0 - 0 + 4X(s) = F(s)$

$X(s)(s^2+4) = F(s)$

$f(t) = \cos 2t\,(1 - u(t-2\pi))$

$= \cos 2t - \cos 2t\, u(t-2\pi)$

$= \cos 2t - \cos(2(t-2\pi))\, u(t-2\pi)$

$F(s) = \frac{s}{s^2+4} - e^{-2\pi s}\frac{s}{s^2+4}$

$X(s) = \frac{1}{s^2+4}\left(\frac{s}{s^2+4} - e^{-2\pi s}\frac{s}{s^2+4}\right)$

$= \frac{s}{(s^2+4)^2} - e^{-2\pi s}\frac{s}{(s^2+4)^2}$

Table!

$x(t) = \frac{1}{4} t \sin 2t - u(t-2\pi)\frac{1}{4}(t-2\pi)\sin 2t$

$= \begin{cases} \frac{t}{4}\sin 2t & 0 \le t < 2\pi \\ \frac{\pi}{2}\sin 2t & t \geq 2\pi \end{cases}$

```
> with(plots):
> plot1:=plot(.25*t*sin(2*t)-Heaviside(t-2*Pi)*(t-2*Pi)/4*sin(2*t),
      t=0..15,color=black):
  plot2:=plot(t/4,t=0..2*Pi,color=black,linestyle=2):
  plot3:=plot(-t/4,t=0..2*Pi,color=black,linestyle=2):
  plot4:=plot(Pi/2,t=2*Pi..15,color=black,linestyle=2):
  plot5:=plot(-Pi/2,t=2*Pi..15,color=black,linestyle=2):
  display({plot1,plot2,plot3,plot4,plot5},title=`resonance
    turned off`);
```