Math 2280-1
Wed 4/13

Return to the "land of linear."

§7.1-7.2 : intro to $\underbrace{\text{Laplace transform}}$

one of a number of interesting linear integral transformations of functions which convert DE or PDE problems into algebra problems, because derivatives of input fcns get converted in multiplication by the output variable (?!).

others: Fourier series, Fourier transform...
↑ chptr 9

particularly suited for DE initial value problems...

Let $f(t)$ be a function defined for $t \geq 0$ s.t. f grows at most exponentially fast ($|f(t)| \leq Ce^{Mt}$ for some C, M)

Then the Laplace transform function $F(s) = \mathcal{L}\{f(t)\}(s)$ is defined for $s > M$ by

$$\boxed{\begin{array}{l}\mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t)\,dt \\ \quad \| \\ F(s)\end{array}}$$

examples

- $\mathcal{L}\{1\}(s) = \int_0^\infty e^{-st}\,1\,dt = \left.\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s} \qquad (s>0)$

- $\mathcal{L}\{e^{\alpha t}\}(s) = \int_0^\infty e^{-st} e^{\alpha t}\,dt = \int_0^\infty e^{(\alpha - s)t}\,dt = \left.\frac{1}{\alpha - s} e^{(\alpha-s)t}\right]_0^\infty = \frac{1}{s-\alpha}$ $\quad s>\alpha$, or $s > Re(\alpha)$ if α is complex.

- Laplace transform is linear!

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\}(s) = \int_0^\infty e^{-st}\left(c_1 f_1(t) + c_2 f_2(t)\right) dt$$
$$= c_1 \int_0^\infty e^{-st} f_1(t)\,dt + c_2 \int_0^\infty e^{-st} f_2(t)\,dt$$
$$= c_1 F_1(s) + c_2 F_2(s)$$

- $\alpha = ik$ imaginary

$$\Rightarrow \mathcal{L}\{e^{ikt}\}(s) = \frac{1}{s-ik} \cdot \frac{s+ik}{s+ik} = \frac{s+ik}{s^2+k^2} = \frac{s}{s^2+k^2} + i\boxed{\frac{k}{s^2+k^2}}$$

|| Euler

$$\mathcal{L}\{\cos kt + i \sin kt\}(s) = \mathcal{L}\{\cos kt\}(s) + i\boxed{\mathcal{L}\{\sin kt\}(s)}$$

- similarly, if $\alpha = a + ik$ then

$$\mathcal{L}\{e^{at}\cos kt + i e^{at} \sin kt\}(s) = \mathcal{L}\{e^{\alpha t}\}(s) = \frac{1}{s-\alpha} = \frac{1}{s-(a+ik)}$$
$$= \frac{1}{(s-a)-ik} \cdot \frac{(s-a)+ik}{(s-a)+ik}$$

$$\mathcal{L}\{e^{at}\cos kt\}(s) + i\boxed{\mathcal{L}\{e^{at}\sin kt\}(s)} = \frac{s-a}{(s-a)^2+k^2} + i\boxed{\frac{k}{(s-a)^2+k^2}}$$

Exercise 1

Compute $\mathcal{L}\{7 - 5e^{2t} + 4\cos 2t\}(s)$

$f(t)$	$F(s)$
1	$1/s$
e^{at}	$\frac{1}{s-a}$
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$
$\cos kt$	$\frac{s}{s^2+k^2}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$\vdots$	

to be continued! (see book cover)

Exercise 2 It's a (difficult to prove) theorem that a given Laplace transform $F(s)$ can arise from at most one continuous function $f(t)$. So, you can use linearity and a good table to compute inverse Laplace transforms.

Find $\mathcal{L}^{-1}\left\{ \frac{4}{s^2+1} - \frac{s}{s^2+4} \right\}(t)$

Laplace transforms and DE's :

$$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty \underbrace{e^{-st}}_{v} \underbrace{f'(t)dt}_{du} = e^{-st} f(t) \Big]_{t=0}^{\infty} - \int_0^\infty -se^{-st} f(t)dt$$

$dv = -se^{-st}dt$

$u = f(t)$

$$= -f(0) + sF(s) \qquad (s > M)$$

$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$\vdots$	

so

$$\mathcal{L}\{f''(t)\}(s) = s\underbrace{\mathcal{L}\{f'(t)\}(s)}_{(-f(0)+sF(s))} - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0)$$

Exercise 3 : Solve IVP with Laplace !! (without x_p, x_H, etc.).

$$* \begin{cases} x'' + 4x(t) = 10\cos 3t \\ x(0) = 2 \\ x'(0) = 1 \end{cases}$$

(non-resonant undamped forced oscillator)

Answer: write $\mathcal{L}\{x(t)\}(s) = X(s)$. We know a unique soln $x(t)$ to $*$ exists. for this soln,

$$\mathcal{L}\{x''(t)\}(s) + 4\mathcal{L}\{x(t)\}(s) = 10\mathcal{L}\{\cos 3t\}(s)$$

$$s^2X(s) - s(2) - 1 + 4X(s) = 10 \cdot \frac{s}{s^2+9}$$

$$X(s)(s^2+4) = \frac{10s}{s^2+9} + 2s + 1$$

$$X(s) = 10s\left(\frac{1}{(s^2+4)(s^2+9)}\right) + \frac{2s+1}{s^2+4}$$

$$= 10s\left(\frac{1}{s^2+4} - \frac{1}{s^2+9}\right)\frac{1}{5} + \frac{2s+1}{s^2+4}$$

$$\frac{4s}{s^2+4} + \frac{1}{s^2+4} - \frac{2s}{s^2+9}$$

so,

$$\boxed{x(t) = 4\cos 2t + \tfrac{1}{2}\sin 2t - 2\cos 3t}$$

Magic!!

(and quicker than $x_p + x_H$)

Exercise 4 Solve an unforced damped oscillator problem

$$\begin{cases} x'' + 6x' + 34x = 0 \\ x(0) = 3 \\ x'(0) = 1 \end{cases}$$

via Laplace transform.

Laplace: $s^2 X(s) - 3s - 1 + 6\left(sX(s) - 3\right) + 34X(s) = 0$

$$X(s)\left(s^2 + 6s + 34\right) = 3s + 1 + 18 = 3s + 19$$

$$X(s) = \frac{3s+19}{s^2+6s+34}$$ ← to use table, complete square in denom, then complete the linear in numerator

$$= \frac{3s+19}{(s+3)^2 + (34-9)}$$

$$= \frac{3s+19}{(s+3)^2 + 25}$$

$$X(s) = \frac{3(s+3)}{(s+3)^2+25} + \frac{10}{(s+3)^2+25}$$

So $\boxed{x(t) = 3e^{-3t}\cos 5t + 2e^{-3t}\sin 5t}$!

in §7.1-7.3 we fill in more of the Laplace transform table entries, solve applied and general IVP's, get very good at partial fractions and at completing the square, and even see some new applications

well, these are mostly in 7.4-7.5, which will be optional.

(can try some 7.1-7.2 HW if time)

Table of Laplace Transforms (front cover of text.)

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 10.

Function	Transform	Function	Transform
$f(t)$	$F(s)$	e^{at}	$\dfrac{1}{s-a}$
$af(t)+bg(t)$	$aF(s)+bG(s)$	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s)-f(0)$	$\cos kt$	$\dfrac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	$\sin kt$	$\dfrac{k}{s^2+k^2}$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	$\cosh kt$	$\dfrac{s}{s^2-k^2}$
$\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$	$\sinh kt$	$\dfrac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	$\dfrac{1}{2k^3}(\sin kt-kt\cos kt)$	$\dfrac{1}{(s^2+k^2)^2}$
$tf(t)$	$-F'(s)$	$\dfrac{t}{2k}\sin kt$	$\dfrac{s}{(s^2+k^2)^2}$
$t^nf(t)$	$(-1)^nF^{(n)}(s)$	$\dfrac{1}{2k}(\sin kt+kt\cos kt)$	$\dfrac{s^2}{(s^2+k^2)^2}$
$\dfrac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\dfrac{e^{-as}}{s}$
$f(t)$, period p	$\dfrac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\dfrac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\dfrac{1}{s}\tanh\dfrac{as}{2}$
t	$\dfrac{1}{s^2}$	$\left[\!\left[\dfrac{t}{a}\right]\!\right]$ (staircase)	$\dfrac{e^{-as}}{s(1-e^{-as})}$
t^n	$\dfrac{n!}{s^{n+1}}$		
$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$		
t^a	$\dfrac{\Gamma(a+1)}{s^{a+1}}$		