

Math 2280-1
Friday March 5

- We rushed through page 4 Wed,
about the correspondence between
 n^{th} order linear DE's and certain
special first order systems of n linear DE's.

Now you've done (hw) A, B, so you've thought
more about this correspondence and also how
the IVP for one corresponds to the IVP for the other.

You have a geometric reason (or physics reason) to believe $\exists!$ thm for
general linear 1^{st} order system IVP's. Thus you deduce why $\exists!$ held for
 n^{th} order linear IVP's in Chapter 3.

- Quickly go through Wed Maple notes to solidify your belief in $\exists!$ for IVP, and
for hints on your Runge-Kutta (hw).

then 4.5.1-4.5.2: Example 1, p 307, 4.5.2:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- What is the dimension of the sol'n space to this system of DE's?

- Find a basis for sol'n space of the form $\{e^{\lambda_1 t} \vec{v}_1, e^{\lambda_2 t} \vec{v}_2\}$.

(recall, for $\vec{x}' = A\vec{x}$, $\vec{x} = e^{\lambda t} \vec{v}$ is a sol'n iff $A\vec{v} = \lambda \vec{v}$).

- What is the Wronskian for your basis above?

Hw for Friday 3/12

(1)

- 4.3 (9) You may hand in joint work
with up to one other person on this
problem, which will require you to
write some Runge-Kutta code for systems

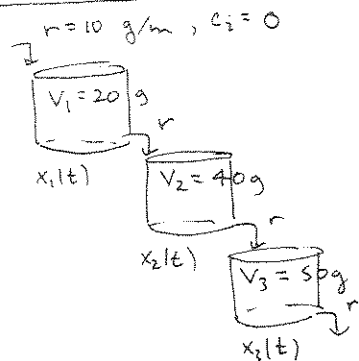
5.1 11, 13, 18 21, 22, 26, 31, 35

5.2 2, 3 9, 15, 27, 35 You can use
Maple (or other software) for evals
& events, and to plot the phase
portraits. (Or use "pplane (google it!)
for the phase portraits.)

5.3 3 7, 9, 14, 16, 17 21

The Maple exploration p 330-332
will be due 3/19

example 2 p.309



$$\frac{dx_1}{dt} = (-10) \frac{x_1}{20} = -.5x_1$$

$$\frac{dx_2}{dt} = 10 \frac{x_1}{20} - 10 \frac{x_2}{40}$$

$$\frac{dx_3}{dt} = 10 \frac{x_2}{40} - 10 \frac{x_3}{50}$$

given $x_1(0) = 15$ (1b)
 $x_2(0) = 0$
 $x_3(0) = 0$

Find $\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} -.5 & 0 & 0 \\ .5 & -.25 & 0 \\ 0 & .25 & -.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Use MAPLE!!
 (knowing you could)
 work by hand

> with(linalg):
 > A:=matrix(3,3,[-.5,0,0,.5,-.25,0,0,.25,-.2]);

(older linear algebra library commands)

$$A := \begin{bmatrix} -.5 & 0 & 0 \\ .5 & -.25 & 0 \\ 0 & .25 & -.2 \end{bmatrix}$$

> eigenvectors(A);
 [-0.5, 1, [[1, -2.000000000, 1.666666667]]], [-0.25, 1, [[0, 1, -5.000000]]],
 [-0.2, 1, [[0, 0, 1]]]

So $\vec{x}(t) = c_1 e^{-.5t} \begin{bmatrix} 3 \\ -6 \\ 25 \end{bmatrix} + c_2 e^{-.25t} \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} + c_3 e^{-.2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\vec{x}(0) = \begin{bmatrix} 3 & 0 & 0 \\ -6 & 1 & 0 \\ 25 & -5 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} c_1 = 5 \\ c_2 = 30 \\ c_3 = 125 \end{matrix} \Rightarrow \vec{x}(t) = 5e^{-.5t} \begin{bmatrix} 3 \\ -6 \\ 25 \end{bmatrix} + 30e^{-.25t} \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} + 125e^{-.2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

> with(LinearAlgebra):
 A := Matrix(3,3,[-1/2,0,0,1/2,-1/4,0,0,1/4,-1/5]);
 Eigenvectors(A);

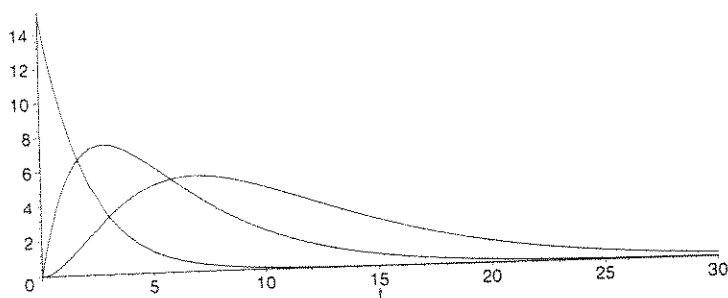
(newer linear algebra library commands; not actually as good.)

$$A := \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{5} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{5} \\ -\frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{5} & 0 & 0 \\ -\frac{6}{5} & 0 & -\frac{1}{5} \\ 1 & 1 & 1 \end{bmatrix}$$

> plot((15*exp(-.5*t), -30*exp(-.5*t)+30*exp(-.25*t),
 25*exp(-.5*t)-150*exp(-.25*t)+125*exp(-.2*t)), t=0..30,
 color=black, title='amounts in each tank');

amounts in each tank

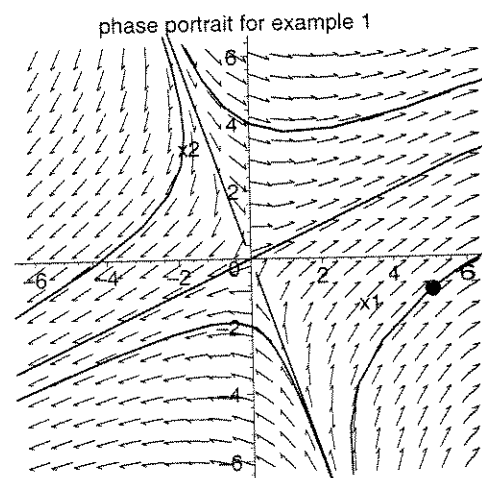


Solve

$$\begin{cases} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \end{cases}$$

$$\vec{x}(t) = e^{-2t} \begin{bmatrix} 1 \\ -3 \end{bmatrix} + 2e^{5t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

```
>
> with(DEtools):
> phaseportrait([diff(x1(t),t)=4*x1(t)+2*x2(t),
diff(x2(t),t)=3*x1(t)-x2(t)),
[x1(t),x2(t)],t=-1..1,
{[x1(0)=5,x2(0)=-1],[x1(0)=0,x2(0)=4],
[x1(0)=2,x2(0)=1],[x1(0)=-1,x2(0)=3],
[x1(0)=-2,x2(0)=-1],[x1(0)=1,x2(0)=-3],
[x1(0)=-3,x2(0)=1],[x1(0)=0,x2(0)=-2]
},x1=-6..6,x2=-6..6,
color=black,linecolor=black,
title='phase portrait for example 1');
```



Theorem 1 If $A_{n \times n}$ is diagonalizable, i.e. has an $\mathbb{R}^n$ basis of eigenvectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ (with corresponding evals $\lambda_1, \dots, \lambda_n$), then

$\{e^{\lambda_1 t} \vec{v}_1, \dots, e^{\lambda_n t} \vec{v}_n\}$ is a basis of solns to $\frac{d\vec{x}}{dt} = A\vec{x}$

this is guaranteed to happen if $\lambda_1, \lambda_2, \dots, \lambda_n$ are distinct

proof: 2270 for the last statement, i.e. $\lambda_1, \dots, \lambda_n$ distinct $\Rightarrow \{\vec{v}_1, \dots, \vec{v}_n\}$ l.i. $\Rightarrow \mathbb{R}^n$ Basis

$\Rightarrow e^{\lambda_1 t} \vec{v}_1, \dots, e^{\lambda_n t} \vec{v}_n$ l.i. $\blacksquare$

(we can recall the proof!)

(there's a cool one using Vandermonde det.)

(or you can assume

$\lambda_1 < \lambda_2 < \dots < \lambda_n$ (or $\text{Re } \lambda_i$)

and argue by limits.)

Example 3: Glucose-insulin model (adapted from a discussion on page 340-341 of the text "Linear Algebra with Applications," by Otto Bretscher)

Let $G(t)$ be the excess glucose concentration (mg of G per 100 ml of blood, say) in someone's blood, at time t hours. Excess means we are keeping track of the difference between current and equilibrium ("fasting") concentrations. Similarly, Let $H(t)$ be the excess insulin concentration at time t . When blood levels of glucose rise, say as food is digested, the pancreas reacts by secreting insulin in order to utilize the glucose. Researchers have developed mathematical models for the glucose regulatory system. Here is a simplified (linearized) version of one such model, with particular representative matrix coefficients. It would be meant to apply between meals, when no additional glucose is being added to the system:

$$\begin{bmatrix} \frac{dG}{dt} \\ \frac{dH}{dt} \end{bmatrix} = \begin{bmatrix} -.1 & -.4 \\ .1 & -.1 \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix}$$

Explain (understand) the signs of the matrix coefficients:

In Bretscher the topic at hand was **discrete dynamical systems** rather than **continuous** ones, and the model given was:

$$\begin{bmatrix} G(t+1) \\ H(t+1) \end{bmatrix} = \begin{bmatrix} .9 & -.4 \\ .1 & .9 \end{bmatrix} \begin{bmatrix} G(t) \\ H(t) \end{bmatrix}$$

Explain how I converted this discrete dynamical system into a related first order system of DE's!
(Notice the matrices are NOT the same!)

Now let's solve the initial value problem, say right after a big meal, when

$$\begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

```
> restart:with(linalg):with(plots):
> A:=matrix(2,2,[-.1,-.4,.1,-.1]);
```

$$A := \begin{bmatrix} -.1 & -.4 \\ .1 & -.1 \end{bmatrix}$$

```
> eigenvects(A);
[-0.1 - 0.2000000000 I, 1, {[ -2.000000000 - 0. I, 0. - 1. I]}],
[-0.1 + 0.2000000000 I, 1, {[ -2.000000000 + 0. I, 0. + 1. I]}]
```

Can you get the same eigenvalues and eigenvectors? Extract a basis for the solution space to his homogeneous system of differential equations from the eigenvector information above:

$$\lambda = -.1 \pm .2i$$

$$\text{for } \lambda = -1 + .2i$$

$$\vec{v} = \begin{bmatrix} -2 \\ \pm i \end{bmatrix}$$

$$e^{\lambda t} \vec{v} = e^{(-1+.2i)t} \begin{bmatrix} -2 \\ i \end{bmatrix} =$$

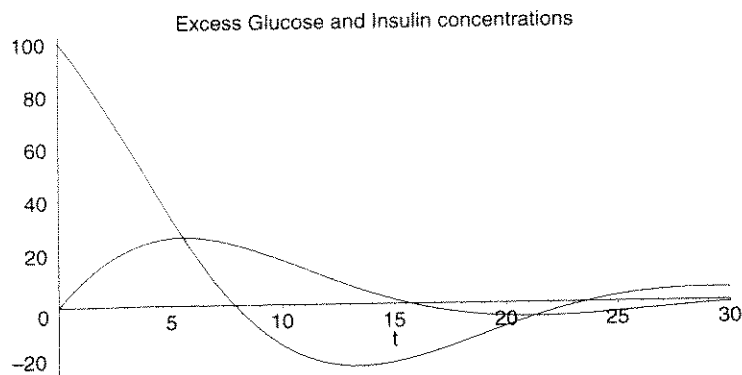
(can you do this
by hand?)

Solve the initial value problem.

Here are some pictures to help understand what the model is predicting:

(1) Plots of glucose vs. insulin, at time t hours later:

```
> G:=t->100*exp(-.1*t)*cos(.2*t):
  H:=t->50*exp(-.1*t)*sin(.2*t):
> plot({G(t),H(t)},t=0..30,color=black,title=
  'Excess Glucose and Insulin concentrations');
```



2) A phase portrait of the glucose-insulin system:

```
> pict1:=fieldplot([-1*G-.4*H,.1*G-.1*H],G=-40..100,H=-15..40):
  soltn:=plot([G(t),H(t),t=0..30],color=black):
  display({pict1,soltn},title='Glucose vs Insulin
  phase portrait');
```

