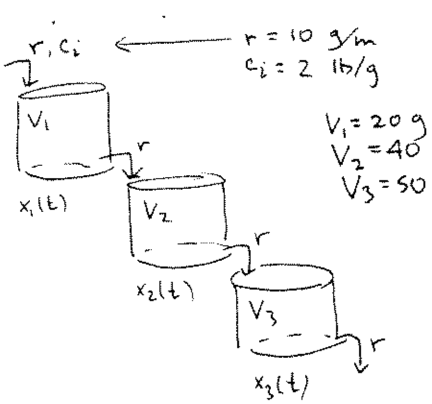
Math 2280-1
Monday March 29

§5.6 Nonhomogeneous linear systems of DE's
(will use e^{At} & FMS's)

$$\frac{d\vec{x}}{dt} = P(t)\vec{x} + \vec{f}(t)$$

- undetermined coeff's in case $P(t) = A$ (const matrix) and $\vec{f}(t)$ is one of good guessing functions
- variation of parameters (☺), for general $P(t)$, $\vec{f}(t)$, provided you already have a FMS $\Phi(t)$ for the homogeneous system $\frac{d\vec{x}}{dt} = P(t)\vec{x}$.

<u>Example</u> p. 363 (Example 2)



$\vec{x}(0) = \vec{0}$.

Explain why the 1st order system for $\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$ is

$$* \quad \frac{d\vec{x}}{dt} = \begin{bmatrix} -.5 & 0 & 0 \\ .5 & -.25 & 0 \\ 0 & .25 & -.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix}$$

general sol'n is

$$\vec{x}(t) = \vec{x}_H(t) + \vec{x}_p(t)$$

as usual.

for $A = \begin{bmatrix} -.5 & 0 & 0 \\ .5 & -.25 & 0 \\ 0 & .25 & -.2 \end{bmatrix}$ Maple says $\lambda = -.5$, $\vec{v} = \begin{bmatrix} 5 \\ -6 \\ 5 \end{bmatrix}$; $\lambda = -.25$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$; $\lambda = -.2$, $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(plus, we analyzed this matrix before!)

So, for $x_H(t)$ a FMS $\Phi(t)$ is

$$\Phi(t) = \begin{bmatrix} \quad \end{bmatrix}$$

and general sol'tn to $\vec{x}_H(t) = \Phi(t)\,\vec{c}$.

For $\vec{x}_p(t)$ we guess a constant vector, since

$$L(\vec{x}) := \vec{x}' - A\vec{x}$$

transforms constant vectors to constant vectors [This is undetermined coeff's easy case for systems!]

If $\vec{x}_p(t) = \vec{k}$, then plug into DE: $\vec{x}' = A\vec{x} + \vec{b}$, page 1

$$\vec{0} = \begin{bmatrix} -.5 & 0 & 0 \\ .5 & -.25 & 0 \\ 0 & .25 & -.2 \end{bmatrix} \vec{k} + \begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix};$$

$$\vec{k} = A^{-1} \begin{bmatrix} -20 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 40 \\ 80 \\ 100 \end{bmatrix} \quad \text{(Maple)}$$

general sol'n

$$\vec{x}(t) = \vec{k} + \Phi(t)\,\vec{c} \qquad (\vec{x}_p + \vec{x}_H), \quad \text{with } \Phi(t) = \left[e^{-.5t}\begin{bmatrix} 5 \\ -6 \\ 5 \end{bmatrix}, e^{-.25t}\begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}, e^{-.2t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right]$$

If $\vec{x}(0) = \vec{0}$ (as in example),

$$\vec{0} = \vec{k} + \Phi(0)\,\vec{c}$$

$$\vec{c} = \Phi(0)^{-1}(-\vec{k}) = \begin{bmatrix} -40/3 \\ -160 \\ -\frac{2500}{3} \end{bmatrix} \quad \text{(Maple)}$$

so $$\vec{x}(t) = \begin{bmatrix} 40 \\ 80 \\ 100 \end{bmatrix} - \frac{40}{3} e^{-.5t} \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix} - 160 e^{-.25t} \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} - \frac{2500}{3} e^{-.2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

```
> x1:=t->40-40*exp(-.5*t);
  x2:=t->80+80*exp(-.5*t)-160*exp(-.25*t);
  x3:=t->100-200/3*exp(-.5*t)+5*160*exp(-.25*t)-
     2500/3*exp(-.2*t);
```

$$x1 := t \to 40 - 40\,e^{(-0.5\,t)}$$

$$x2 := t \to 80 + 80\,e^{(-0.5\,t)} - 160\,e^{(-0.25\,t)}$$

$$x3 := t \to 100 - \frac{200}{3}\,e^{(-0.5\,t)} + 800\,e^{(-0.25\,t)} - \frac{2500}{3}\,e^{(-0.2\,t)}$$

```
> plot({x1(t),x2(t),x3(t)},t=0..30,color=black,title=`three
  tanks becoming salty`);
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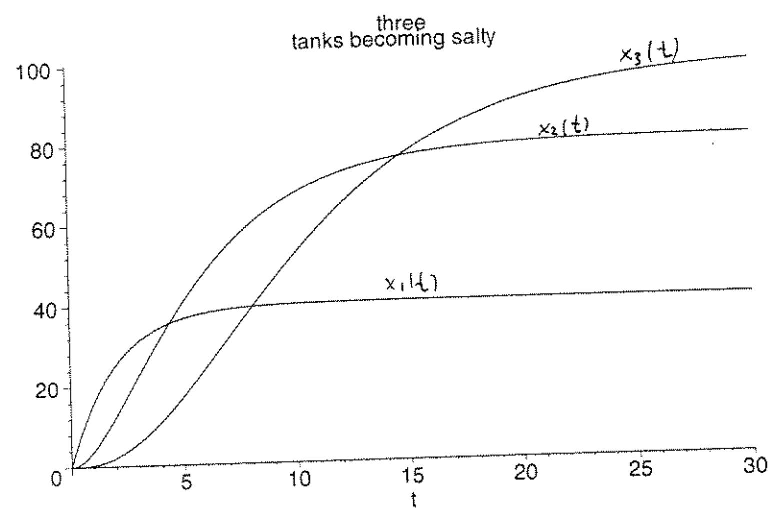


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>
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but, for harder nonhomogeneous problems, you'll prefer to use variation of parameters! ☺

Variation of parameters, revisited !!

Consider the non-homogeneous linear system of DE's

$* \quad \vec{x}'(t) = P(t)\vec{x} + \vec{f}(t)$

(In practice, in this class, $P(t)$ is a constant matrix "A", but the method is more general).

Let $\Phi(t)$ be a FMS for the homogeneous system

$$\vec{x}'(t) = P(t)\vec{x}$$

Since $\Phi(t)$ is invertible $\forall t \in I$, any solution $\vec{x}(t)$ to $*$ can be written as

$$\vec{x}(t) = \Phi(t)\vec{u}(t) \qquad (\text{i.e. } \vec{u} = \Phi^{-1}\vec{x})$$

plug into $*$:

$$\underbrace{\Phi'}_{\substack{\shortparallel \\ P\cancel{\Phi}}}\cancel{\vec{u}} + \Phi\vec{u}' = P\cancel{\Phi}\vec{u} + \vec{f}$$

i.e. $\Phi\vec{u}' = \vec{f}$, i.e. $\boxed{\vec{u}' = \Phi^{-1}\vec{f}}$; $\boxed{\vec{u} = \int \Phi^{-1}\vec{f}\,dt}$

<u>Exercise 1</u> (See p. 368 for case $n=2$)

When we were doing higher order linear eqtns, we considered the DE (Feb 22)

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = f$$

For a basis $\{y_1, \dots y_n\}$ of homogeneous sol'ns the var. par. formula for y_p was

$$y_p = u_1y_1 + u_2y_2 + \dots + u_ny_n$$

where $\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = [W]^{-1}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f \end{bmatrix}$

Explain why this formula (which seemed especially mysterious at the time) is actually a special case of the boxed formula above!

If you want to solve the IVP

$$\begin{cases} \vec{x}'(t) = P(t)\vec{x} + \vec{f}(t) \\ \vec{x}(0) = \vec{x}_0 \end{cases}$$

Then $\vec{x} = \Phi \vec{u}$ for $\vec{u}' = \Phi^{-1} f$

$\vec{x}_0 = \Phi(0)\vec{u}_0$

so $\vec{u}_0 = \Phi(0)^{-1}\vec{x}_0$

Thus $\vec{u}(t) = \vec{u}_0 + \int_0^t \vec{u}'(s)ds$

$\vec{u}(t) = \vec{u}_0 + \int_0^t \Phi^{-1}(s)\vec{f}(s)ds$

$\vec{x}(t) = \Phi(t)\vec{u}(t)$

$$\boxed{\vec{x}(t) = \Phi(t)\left[\Phi(0)^{-1}\vec{x}_0 + \int_0^t \Phi^{-1}(s)\vec{f}(s)ds\right]}$$

If you want to solve the special case IVP $(P(t)=A$, const.$)$

$$\begin{cases} \vec{x}'(t) = A\vec{x} + \vec{f}(t) \\ \vec{x}(0) = \vec{x}_0 \end{cases}$$

mimic chapter 1!

$$\vec{x}' - A\vec{x} = \vec{f}$$

$$e^{-At}(\vec{x}' - A\vec{x}) = e^{-At}\vec{f}(t)$$

$$(e^{-At}\vec{x})' = e^{-At}\vec{f}$$

$\int_0^t$: $\quad e^{-At}\vec{x}(t) - \vec{x}_0 = \int_0^t e^{-As}\vec{f}(s)ds$

$$e^{-At}\vec{x}(t) = \vec{x}_0 + \int_0^t e^{-As}\vec{f}(s)ds$$

$(e^{-At})^{-1} = e^{At}$! (why??)

So

$$\boxed{\vec{x}(t) = e^{At}\vec{x}_0 + e^{At}\int_0^t e^{-As}\vec{f}(s)ds}$$

(is a special case of box on left).

Exercise 2 (p. 367.)

Solve
$$\begin{cases} \vec{x}\,' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \vec{x} - \begin{bmatrix} 15 \\ 4 \end{bmatrix} t e^{-2t} \\ \vec{x}(0) = \begin{bmatrix} 7 \\ 3 \end{bmatrix} \end{cases}$$

using $\vec{x}(t) = e^{At}\vec{x}_0 + e^{At}\int_0^t e^{-As}\vec{f}(s)\,ds$

(and technology help!)

```
> with(LinearAlgebra):
> A := Matrix(2, 2, [4, 2, 3, -1]);
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$$A := \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \qquad (1)$$

```
> Eigenvectors(A);
```

$$\begin{bmatrix} 5 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 & -\frac{1}{3} \\ 1 & 1 \end{bmatrix} \qquad (2)$$

$\lambda = 5$ $\qquad\qquad$ $\lambda = -2$

$\vec{v} =$ $\qquad\qquad$ $\vec{v} =$

So, if you were using method of undetermined coeff's what would be your guess

$\vec{x}_p(t) =$

> $x0 := Vector([7, 3]);$
> $f := t \to t \cdot \exp(-2 \cdot t) \cdot Vector([15, 4]);$

$$x0 := \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$f := t \to t\, e^{-2t}\, Vector([15, 4]) \tag{3}$$

> $f(s);$ *#just checking*

$$\begin{bmatrix} 15\, s\, e^{-2s} \\ 4\, s\, e^{-2s} \end{bmatrix} \tag{4}$$

> $MatrixExponential(A, -s).f(s);$ *#my integrand*

$$\begin{bmatrix} 15 \left(\frac{1}{7} e^{2s} + \frac{6}{7} e^{-5s} \right) s\, e^{-2s} + 4 \left(\frac{2}{7} e^{-5s} - \frac{2}{7} e^{2s} \right) s\, e^{-2s} \\ 15 \left(\frac{3}{7} e^{-5s} - \frac{3}{7} e^{2s} \right) s\, e^{-2s} + 4 \left(\frac{6}{7} e^{2s} + \frac{1}{7} e^{-5s} \right) s\, e^{-2s} \end{bmatrix} \tag{5}$$

> $integrand := map(simplify, \%);$

$$integrand := \begin{bmatrix} s \left(1 + 14\, e^{-7s} \right) \\ s \left(7\, e^{-7s} - 3 \right) \end{bmatrix} \tag{6}$$

>

> $int\left(s \left(1 + 14\, e^{-7s}\right), s = 0..t\right);$
> $int\left(s \left(7\, e^{-7s} - 3\right), s = 0..t\right);$

$$\frac{2}{7} + \frac{1}{2} t^2 - 2\, t\, e^{-7t} - \frac{2}{7} e^{-7t}$$

$$\frac{1}{7} - t\, e^{-7t} - \frac{1}{7} e^{-7t} - \frac{3}{2} t^2 \tag{7}$$

> $x := t \to MatrixExponential(A, t).x0 + MatrixExponential(A, t).Vector\left(\left[\frac{2}{7} + \frac{1}{2} t^2 - 2\, t\, e^{-7t} - \frac{2}{7} e^{-7t}, \frac{1}{7} - t\, e^{-7t} - \frac{1}{7} e^{-7t} - \frac{3}{2} t^2\right]\right):$

> $simplify(x(t));$

$$\begin{bmatrix} -\frac{1}{7} e^{-2t} + \frac{50}{7} e^{5t} + \frac{1}{2} e^{-2t} t^2 - 2\, e^{-2t}\, t \\ \frac{25}{7} e^{5t} - \frac{4}{7} e^{-2t} - e^{-2t}\, t - \frac{3}{2} e^{-2t} t^2 \end{bmatrix} \tag{8}$$

← compare to page 368

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