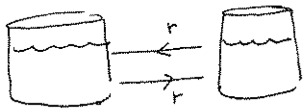
Math 2280-1
Tuesday March 2
§4.1: Introduction to systems of differential equations

<u>example</u>: tank systems (§5.2)



$V_1 = 100$ g $\qquad V_2 = 50$ g
$x_1(t)$ solute $\qquad x_2(t)$

$$\frac{dx_1}{dt} = -10\frac{x_1}{100} + 10\frac{x_2}{50}$$

$$\frac{dx_2}{dt} = 10\frac{x_1}{100} - 10\frac{x_2}{50}$$

in matrix form, and IVP:

$$\begin{cases} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -.1 & .2 \\ .1 & -.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \underset{=}{\text{e.g.}} \begin{bmatrix} 15 \\ 15 \end{bmatrix} \end{cases}$$

This is an example of a 1st order, constant coefficient, homogeneous linear system of DE's!

$$\text{IVP}\begin{cases} \vec{x}'(t) = A\vec{x} \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

- Sol'n space will be n-dim'l **vector space**
  - closed wrt + & scalar mult
  - T: sol's $\vec{x}(t)$ $\to$ $\vec{x}(t_0) \in \mathbb{R}^n$ is isomorphism!

- look for basis of sol'ns of the form

$$\vec{x}(t) = e^{\lambda t}\vec{v}$$

(analogy to scalar 1st order linear homog DE)

$$\Rightarrow \vec{x}'(t) = \lambda e^{\lambda t}\vec{v} \qquad (\vec{v} \text{ is const})$$

if also $\vec{x}'(t) = A\vec{x}$

deduce $\quad A e^{\lambda t}\vec{v} = \lambda e^{\lambda t}\vec{v}$

$$\overset{\shortparallel}{\cancel{e^{\lambda t}}A\vec{v}}$$

holds iff $A\vec{v} = \lambda\vec{v}$
($\vec{v}$ is an eigenvector for A with eigenvalue λ!)

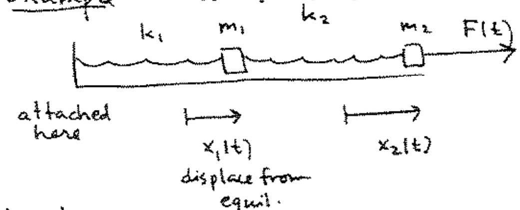
(so if A is diagonalizable we'll get a basis of sol'ns $\{e^{\lambda_1 t}\vec{v}_1, \ldots e^{\lambda_n t}\vec{v}_n\}$!)

do you already dare to solve the IVP on page 1?

ans: $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \end{bmatrix} - 5e^{-.3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

makes sense!!
(even to your little sister)

<u>Example</u>: mass-spring systems (§5.3)



Newton:

$$m_1 x_1''(t) = k_2 \underbrace{(x_2 - x_1)}_{\text{2nd spring is stretched this much}} - k_1 x_1$$

$$m_2 x_2''(t) = -k_2(x_2 - x_1) + F(t)$$

e.g. $\left.\begin{array}{l} m_1 = 2 \\ m_2 = 1 \\ k_1 = 4 \\ k_2 = 2 \\ f(t) = 40\sin 3t \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 2x_1'' = 2(x_2 - x_1) - 4x_1 = -6x_1 + 2x_2 \\ x_2'' = -2(x_2 - x_1) + 40\sin 3t = 2x_1 - 2x_2 + 40\sin 3t \end{array}\right.$

or $\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 40\sin 3t \end{bmatrix}$

- What's a natural IVP for this 2nd order system of 2 linear DE's?
  (We won't solve it today – see §7.4)

<u>note</u>: Any system of DE's can be converted into an equivalent system of 1st order DE's, by introducing extra unknown functions.

(1) $\boxed{\begin{array}{l} x_1'' = -3x_1 + x_2 \\ x_2'' = 2x_1 - 2x_2 + 40\sin 3t \end{array}}$

Let $y_1 = x_1'$

$y_2 = x_2'$

(2) $\boxed{\begin{array}{l} x_1' = y_1 \\ y_1' = -3x_1 + x_2 \\ x_2' = y_2 \\ y_2' = 2x_1 - 2x_2 + 40\sin 3t \end{array}}$

solutions of (1) yield solutions of (2) <u>and</u> vise verse. CHECK!!

This correspondence is sometimes useful.

We'll solve systems like this in §5.3

Example: restudy a 2^{nd} order (mass-spring) DE
by converting it into a 1^{st} order system of 2 DE's.

2nd order linear homog. DE

$$x'' + .2x' + 1.01x = 0$$

$$x = e^{rt}$$

$$L(x) = e^{rt}\left(\underbrace{r^2 + .2r + 1.01}\right)$$

$$(r+.1)^2 + 1$$

$$(r+.1+i)(r+.1-i) = 0; \quad r = -.1 \pm i$$

$$x_H(t) = e^{-.1t}\left(A\cos t + B\sin t\right)$$

$$= e^{-.1t}\left(C\cos(t-\alpha)\right)$$

1'st order system of 2 DE's

$$y := x'$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -1.01x - .2y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1.01 & -.2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

we'll use the work on the left to solve this system, although it is fun to start the eigenvalue-eigenvector method:

$$\begin{vmatrix} 0-\lambda & 1 \\ -1.01 & -.2-\lambda \end{vmatrix} = \lambda^2 + .2\lambda + 1.01$$

DOES THIS LOOK FAMILIAR ???

but, rather than dealing with complex eigenvectors today, steal the solution from left column!

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_H \\ x_H' \end{bmatrix}$$

$$= \underbrace{Ce^{-.1t}}_{\text{spirals inward to } \vec{0} \text{ as } t\to\infty} \underbrace{\begin{bmatrix} \cos(t-\alpha) \\ -.1\cos(t-\alpha) - \sin(t-\alpha) \end{bmatrix}}_{\text{lives on the ellipse}}$$

spirals inward to $\vec{0}$ as $t \to \infty$.

lives on the ellipse $x^2 + (y - .1x)^2 = 1$

i.e. $1.01x^2 - .2xy + y^2 = 1$

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>
> with(plots):
Warning, the name changecoords has been redefined
> pict1:=fieldplot([y,-1.01*x-.2*y],x=-1..1,y=-1..1):
  soltn:=plot([exp(-.1*t)*cos(t),exp(-.1*t)*(-.1*cos(t)-sin(t)),t=0.
  .400],color=black):
  display({pict1,soltn},title=`phase portrait for a damped spring`);
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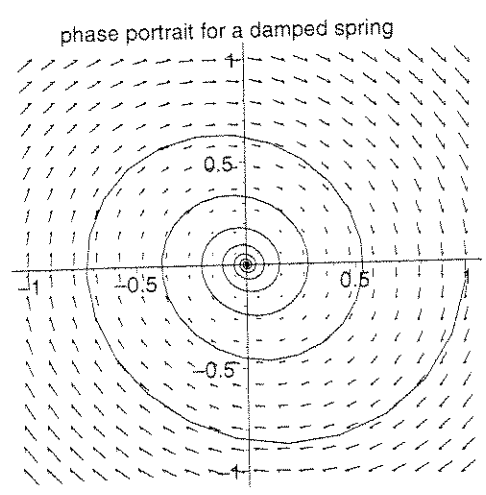


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>
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"phase-space picture" for damped mass spring (chapter 6)
x = position