Math 2280-1
Friday March 19

§5.5 examples and discussion of matrix exponentials and other fundamental matrix solns.

HW for Wednesday March 31
(exam 2 is Friday April 2)

§5.6 1, 13, 15, 19, 23
Do 13, 15 by hand. On 19, 23 you may use technology to help compute variation of parameters formulas for the solutions

- Discuss the universal product rule, which will be very useful (and has been already). ~ page 3 Wednesday.

- Then compute e^{Nt} for the nilpotent matrix N from last example on Wednesday:

$$N = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad N^2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad N^3 = \begin{bmatrix} 0 \end{bmatrix}$$

so N is nilpotent ($N^k = 0$ some k)

thus
$$e^{Nt} = I + Nt + N^2\frac{t^2}{2!} + [0] = \begin{bmatrix} 1 & t & 2t + t^2/2! \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix}$$

If $A = \lambda I + N$ (N nilpotent)

then
$$e^{At} = e^{(\lambda I + N)t} = e^{\lambda I t} e^{Nt} = e^{\lambda t} I e^{Nt} = e^{\lambda t} e^{Nt}$$

because λI and N commute

since exponentials of diagonal matrices are diagonal matrices of exponentials

So if $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix} = 3I + N$. (as above)

$$\boxed{e^{At} = e^{3t} e^{Nt} = e^{3t}\begin{bmatrix} 1 & t & 2t + t^2/2! \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix}}$$

Chains approach, same A

$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & & \ast \\ 0 & 3-\lambda & \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^3 ;$$

$\lambda = 3$ is only eigenvalue.
alg. mult is 1, since

$$[A-3I \,\vdots\, 0] = [N \,\vdots\, 0] = \left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\downarrow \text{rref}$$

$$\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \qquad \begin{array}{l} v_3 = 0 \\ v_2 = 0 \\ v_1 = s \end{array} \qquad \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

E_3 is only one-dim'l.

But by (to be proven maybe next week) theorem,

$$G_3 := \ker(A-3I)^{3} \leftarrow \text{alg. mult here}$$

has $\dim(G_3) = $ alg mult $= 3$.

Thus $G_3 = \mathbb{R}^3$.

We search for a chain

$$\begin{array}{ccc} A-3I & (A-3I)^2 & (A-3I)^3 \\ \begin{array}{ccc} 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{array} & \begin{array}{ccc} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} & 0 \end{array}$$

$\vec{v}_3 = \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ sats $(A-3I)^2 \vec{v}_3 \neq \vec{0}$

$\vec{v}_2 = (A-3I)\vec{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

$\vec{v}_1 = (A-3I)\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$; $\left((A-3I)\vec{v}_1 = \vec{0}\right)$

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ chain of length 3.

yields FMS

$$\Phi(t) = \left[\, e^{3t}\vec{v}_1 \;\middle|\; e^{3t}(t\vec{v}_1 + \vec{v}_2) \;\middle|\; e^{3t}\left(\tfrac{t^2}{2}\vec{v}_1 + t\vec{v}_2 + \vec{v}_3\right) \right] = e^{3t}\begin{bmatrix} -1 & -t+2 & -\frac{t^2}{2}+2t \\ 0 & -1 & -t \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Phi(0) = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Phi(0)^{-1} = \begin{bmatrix} -1 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{At} = \Phi(t)\Phi(0)^{-1}$$

$$= e^{3t}\begin{bmatrix} -1 & -t+2 & -\frac{t^2}{2}+2t \\ 0 & -1 & -t \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= e^{3t}\begin{bmatrix} 1 & t & -t^2/2 + 2t \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix}$$

agrees with page 1

<u>Chain-free way to get $\Phi(t)$:</u> $|A-\lambda I|$ has factor $(\lambda-\lambda_j)^{k_j}$ with k_j = alg mult

Then $G_{\lambda_j} = \ker(A-\lambda_j)^{k_j}$ has dim k_j

pick a basis $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_{k_j}$ for this generalized eigenspace. (In practice, Maple might help.)

Then $\vec{x}_\ell(t) = e^{At}\vec{v}_\ell$ <u>is</u> the ~~a~~ soltn to $\begin{cases} \vec{x}\,'(t) = A\vec{x} \\ \vec{x}(0) = \vec{v}_\ell \end{cases}$

$$= e^{\lambda_j I t} e^{(A-\lambda_j I)t}\vec{v}_\ell$$

$$= e^{\lambda_j t}\left[e^{(A-\lambda_j I)t}\vec{v}_\ell\right]$$

$$= e^{\lambda_j t}\left[I + (A-\lambda_j I)t + \ldots\right]\vec{v}_\ell$$

$$= e^{\lambda_j t}\left[\vec{v}_\ell + t(A-\lambda_j I)\vec{v}_\ell + \frac{t^2}{2}(A-\lambda_j I)^2\vec{v}_\ell + \ldots \frac{t^{k-1}}{(k-1)!}(A-\lambda_j I)^{k-1}\vec{v}_\ell + \vec{0}\right]$$

finite sum! $\qquad \vec{v}_\ell \in \ker(A-\lambda_j I)^k$!

this gives k_j soltns "for" λ_j
amalgamate to get $\Phi(t)$!

<u>example con't</u> $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix}$ (still) $\quad G_3$ has dim 3, i.e. is $\mathbb{R}^3$
so, pick $\vec{v}_1 = \vec{e}_1$, $\vec{v}_2 = \vec{e}_2$, $\vec{v}_3 = \vec{e}_3$.

$$\vec{x}_1(t) = e^{At}\vec{e}_1 = e^{3t}e^{(A-3I)t}\vec{e}_1 = e^{3t}\Big[I\vec{e}_1 + \underbrace{(A-3I)t\,\vec{e}_1}_{\text{already zero}} + \ldots\Big] = e^{3t}\begin{bmatrix}1\\0\\0\end{bmatrix}.$$

$$\vec{x}_2(t) = e^{At}\vec{e}_2 = e^{3t}e^{(A-3I)t}\vec{e}_2 = e^{3t}\Big[I\vec{e}_2 + \begin{bmatrix}0&1&2\\0&0&-1\\0&0&0\end{bmatrix}t\vec{e}_2 + \underbrace{\begin{bmatrix}0&0&-1\\0&0&0\\0&0&0\end{bmatrix}\tfrac{t^2}{2}\vec{e}_2}_{\text{already zero}} + \cdots\Big]$$

$$= e^{3t}\begin{bmatrix}t\\1\\0\end{bmatrix}$$

$$\vec{x}_3(t) = e^{At}\vec{e}_3 = e^{3t}e^{(A-3I)t}\vec{e}_3$$

$$= e^{3t}\Big[I\vec{e}_3 + \underbrace{\begin{bmatrix}0&1&2\\0&0&-1\\0&0&0\end{bmatrix}t\vec{e}_3}_{\begin{bmatrix}2t\\-t\\0\end{bmatrix}} + \underbrace{\begin{bmatrix}0&0&-1\\0&0&0\\0&0&0\end{bmatrix}\tfrac{t^2}{2}\vec{e}_3}_{\begin{bmatrix}-t^2/2\\0\\0\end{bmatrix}} + \vec{0}\Big]$$

$$= e^{3t}\begin{bmatrix}2t - t^2/2\\-t\\1\end{bmatrix}$$

$$\Phi(t) = \Big[\vec{x}_1\Big|\vec{x}_2\Big|\vec{x}_3\Big] = e^{3t}\begin{bmatrix}1 & t & 2t-t^2/2\\0 & 1 & -t\\0 & 0 & 1\end{bmatrix}$$; also equals e^{At}, in this case

$$\boxed{\text{Relating power series and FMS ways of finding } e^{At}, \text{ in general.}}$$

- easy case, is if A is diagonalizable (over $\mathbb{R}^n$ or $\mathbb{C}^n$).

if $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is an eigenbasis, $A\vec{v}_i = \lambda_i \vec{v}_i$

(1) $$\Phi(t) = \left[e^{\lambda_1 t}\vec{v}_1 \,\Big|\, e^{\lambda_2 t}\vec{v}_2 \,\Big|\, \cdots e^{\lambda_n t}\vec{v}_n \right]$$

Write $S = [\vec{v}_1 | \vec{v}_2 | \cdots \vec{v}_n]$, $S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

Then $\boxed{e^{At} = \Phi(t)\,\Phi(0)^{-1} = \Phi(t)\, S^{-1}}$,

so $e^{At} = S\, e^{\Lambda t} S^{-1}$

(2) Alternately, with power series

$$A = S \Lambda S^{-1}$$

$$\text{so } e^{tA} = I + t(S\Lambda S^{-1}) + \frac{t^2}{2!} S\Lambda^2 S^{-1} + \cdots$$

$$= S\, e^{t\Lambda} S^{-1}$$

$$= \underbrace{[\vec{v}_1 | \cdots \vec{v}_n] \begin{bmatrix} e^{t\lambda_1} & & & 0 \\ & e^{t\lambda_2} & & \\ & & \ddots & \\ 0 & & & e^{t\lambda_n} \end{bmatrix}}_{\Phi(t)} S^{-1}$$

agrees with boxed formula.

- hard case : defective eigenspaces. If you construct bases of G_{λ_j} made out of chains and amalgamate these into the columns of S, then Λ gets replaced by the Jordan Canonical form of A, and an analogous discussion ensues.

Example (p. 2) $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix}$, $S = [\vec{v}_1 | \vec{v}_2 | \vec{v}_3] = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$S^{-1}AS = J = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$\uparrow$
$[T(\vec{v}_3)]_\beta$

$J = 3I + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 3I + \tilde{N}$. $\tilde{N}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\tilde{N}^3 = 0$

$$e^{Jt} = e^{(3I + \tilde{N})t} = e^{3t} I\, e^{\tilde{N}t} = e^{3t} \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$A = SJS^{-1}$

$$e^{At} = S\, e^{Jt} S^{-1}$$

$$= e^{3t} \begin{bmatrix} -1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} S^{-1}$$