Math 2280-1
Wednesday March 17

Recall,
for the 1st order system

$*\quad \vec{x}' = A\vec{x} \qquad A_{n\times n}$ const matrix

- $\Phi(t)$ is a FMS (or FSM if I'm dyslexic)

  means
  $\begin{cases} \Phi' = A\Phi \\ \Phi(t) \text{ invertible (iff } \Phi(0) \text{ invertible)} \end{cases}$ <u>equivalent to</u> the columns of $\Phi(t)$ are a basis for sol'ns to $*$

- If $\Phi(t)$ is a FMS, so is $\Phi(t)C$ whenever C^{-1} exists.

- Sol'tn to IVP
  $\begin{cases} \vec{x}' = A\vec{x} \\ \vec{x}(0) = \vec{x}_0 \end{cases}$ is $\vec{x}(t) = \Phi(t)\left(\Phi^{-1}(0)\,\vec{x}_0\right)$

- $e^{At} :=$ <u>the</u> FMS solving
  $\begin{cases} X' = AX \\ X(0) = I \end{cases}$ (after all, for scalars, e^{at} solves $\begin{cases} x'(t) = ax \\ x(0) = 1 \end{cases}$)
  (notice $col_j(e^{At})$ solves $\begin{cases} \vec{x}_j{}' = A\vec{x}_j \\ \vec{x}_j(0) = \vec{e}_j \end{cases}$, so is unique)

- Thus $e^{At} = \Phi(t)\Phi^{-1}(0)$ for any FMS $\Phi(t)$

- now finish the extended example from Tuesday's notes, and complete p. 3-5.
- then try examples on page 2 of today's notes

Sometimes the power series is easiest:

<u>example 2</u> $A = \Lambda = \begin{bmatrix} \lambda_1 & & O \\ & \lambda_2 & \\ O & \ddots & \lambda_n \end{bmatrix}$; $\Lambda^2 = \begin{bmatrix} \lambda_1^2 & & O \\ & \lambda_2^2 & \\ O & \ddots & \lambda_n^2 \end{bmatrix}$; $\Lambda^k = \begin{bmatrix} \lambda_1^k & & O \\ & \lambda_2^k & \\ O & \ddots & \lambda_n^k \end{bmatrix}$

so $e^{At} = I + t\Lambda + \frac{t^2}{2!}\Lambda^2 + \cdots$

$$= \begin{bmatrix} 1 + t\lambda_1 + \frac{t^2}{2!}\lambda_1^2 + \cdots & & \\ & 1 + t\lambda_2 + \frac{t^2}{2!}\lambda_2^2 + \cdots & \\ O & & \ddots \quad 1 + t\lambda_n + \cdots \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & O \\ & e^{\lambda_2 t} & \\ O & \ddots & e^{\lambda_n t} \end{bmatrix}$$

<u>example 3</u>

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

power series way

$A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$

could you do this with a FMS?

$\Rightarrow A^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $A^4 = I$

$$e^{At} = I + tA + \frac{t^2}{2!}(-I) + \frac{t^3}{3!}(-A) + \frac{t^4}{4!}I + \frac{t^5}{5!}A + \cdots$$

$$= \cos t\, I + \sin t\, A$$

$$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \quad !!$$

<u>example 4</u> $[N]$ is called <u>nil</u> <u>potent</u> if $N^k = 0$ for some k.

then $e^{Nt} = I + tN + \cdots + \frac{t^{k-1}}{(k-1)!}N^{k-1} + 0$!

<u>example</u> $N = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $N^2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $N^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$e^{Nt} = I + tN + \frac{t^2}{2}N^2 = \begin{bmatrix} 1 & t & 2t - t^2/2 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix}$ ← relates to our discussion chains! Explain!

If $A = \lambda I + N$ where N is nilpotent, then

$$e^{At} = e^{(\lambda I + N)t}$$
$$= \underbrace{e^{\lambda I t}}_{e^{\lambda t} I} e^{Nt}$$

$[\, e^{A+B} = e^A e^B$ whenever A and B commute; see HW$]$

$e^{\lambda t} I \; e^{Nt}$

(example 2)

e.g. $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow e^{At} = e^{3t}\begin{bmatrix} 1 & t & 2t - t^2/2 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix}$! ← also related to chains!

Universal product rule

Recall the 1-var. proof of the product rule, write $*$ for "times",
and see the generality in which it really applies: (t is a real variable)

$$D_t(f*g)(t) = \lim_{\Delta t\to 0}\left[f(t+\Delta t)*g(t+\Delta t) - f(t)*g(t)\right]\frac{1}{\Delta t}$$

Def. of deriv

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big[[f(t+\Delta t)-f(t)]*g(t+\Delta t) + f(t)*[g(t+\Delta t)-g(t)]\Big]$$

mult $*$ distributes over addition on both sides

$$= \lim_{\Delta t\to 0}\left[\frac{1}{\Delta t}(f(t+\Delta t)-f(t))\right]*g(t+\Delta t)$$
$$+ \lim_{\Delta t\to 0} f(t)*\left[\frac{1}{\Delta t}(g(t+\Delta t)-g(t))\right]$$

scalar mult distributes into either factor of the product

$$\boxed{D_t(f*g)(t) = f'(t)*g(t) + f(t)*g(t)}$$

limit of product is product of limits.
& differentiable fcns are continuous

__examples__ function times fcn (1210)
dot product, cross product } 2210
scalar fcn times vector fcn, matrix fcn times vector fcn, matrix fcn times matrix fcn } 2280

__not an example__ composition, $f\circ g$!
(which is why we have the chain rule.)