

Math 2280-1
Tuesday March 16

§ 5.5 Matrix exponentials & linear systems of DE's
(first finish the damped oscillating train example from § 5.4)

Def: Consider

$$* \quad \vec{x}' = A \vec{x}$$

$A_{n \times n}$ constant matrix

If $\{\vec{x}_1(t), \vec{x}_2(t), \dots, \vec{x}_n(t)\}$ is a basis of sol's to *

then

$$\Phi(t) = \left[\begin{array}{c|c|c} \vec{x}_1(t) & \dots & \vec{x}_n(t) \end{array} \right] \quad \text{is called a } \left. \begin{array}{l} \text{fundamental} \\ \text{matrix} \\ \text{solution} \end{array} \right\} \text{FMS}$$

Notice, this is equivalent to saying $X(t) = \Phi(t)$ solves

$$\left\{ \begin{array}{l} X' = A X_{n \times n} \\ X(0) \text{ non-singular (i.e. invertible, rank } n, \det \neq 0) \end{array} \right. \quad (\text{just look column by column!})$$

Example 1 p. 350

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 10 = (\lambda-5)(\lambda+2)$$

$$\lambda = 5$$

$$\lambda = -2$$

$$\begin{array}{cc|c} -1 & 2 & 0 \\ 3 & -6 & 0 \end{array}$$

$$\begin{array}{cc|c} 6 & 2 & 0 \\ 3 & 1 & 0 \end{array}$$

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\vec{x}_1(t) = e^{5t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \vec{x}_2(t) = e^{-2t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

possible $\Phi(t) = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix}$

Theorem If $\Phi(t)$ is a FMS for $*$ then the (unique) sol'n to

$$\text{IVP} \quad \begin{cases} \vec{x}' = A\vec{x} \\ \vec{x}(0) = \vec{x}_0 \end{cases}$$

$$\text{is } \vec{x}(t) = \Phi(t) [\Phi^{-1}(0) \vec{x}_0]$$

pf : the general sol'n to $*$ is

$$\vec{x}_H(t) = \Phi(t) \vec{c} \quad (= \{c_1 \vec{x}_1 + c_2 \vec{x}_2 + \dots + c_n \vec{x}_n\})$$

so the $\vec{x}(t) = \Phi(t) [\Phi^{-1}(0) \vec{x}_0]$ solves $*$

$$\text{and, } \vec{x}(0) = [\Phi(0) \Phi^{-1}(0)] \vec{x}_0 = I \vec{x}_0 = \vec{x}_0$$

so $\vec{x}(t)$ solves IVP $\blacksquare$

Example 1 cont'd

solve

$$\begin{cases} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{cases}$$

using page 1 $\Phi(t)$ and theorem above.

$$\text{ans: } \vec{x}(t) = \frac{3}{7} e^{-2t} \begin{bmatrix} 1 \\ -3 \end{bmatrix} + \frac{2}{7} e^{5t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Remark: If $\Phi(t)$ is a FMS and C is invertible,

then so is $\Phi(t)C$ (note - you multiply by C on the right)

since

$$\begin{aligned}\frac{d}{dt}(\Phi(t)C) &= \Phi'(t)C \\ &= (A\Phi)C \\ &= A(\Phi C)\end{aligned}$$

and $\Phi(0)C$ is invertible.

$\Phi(t)\Phi'(0)$ is the best FMS because it is the unique soln to

$$\begin{cases} \frac{dX}{dt} = AX \\ X(0) = I \end{cases}$$

↑ because $\text{col}_j(X)$ is the unique soln to

$$\begin{cases} \frac{d\vec{x}}{dt} = A\vec{x} \\ \vec{x}(0) = \vec{e}_j \end{cases}$$

It's so special we will call it e^{At} !

Notice, the sol'n to the IVP

$$\begin{cases} \frac{d\vec{x}}{dt} = A\vec{x} \\ \vec{x}(0) = \vec{z} \end{cases}$$

is $e^{At}\vec{z}$, "just" like for the Chapter 1 scalar eqn.

Example 1 cont'd

$$e^{At} = \Phi(t)\Phi(0)^{-1} = \begin{bmatrix} e^{2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \frac{1}{7} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} e^{-2t} + 6e^{5t} & -2e^{-2t} + 2e^{5t} \\ -3e^{-2t} + 3e^{5t} & 6e^{-2t} + e^{5t} \end{bmatrix}$$

But wait!

Didn't you like how we derived Euler's formula?

Here's a different (?) way to define e^{At} :

$$A_{n \times n} \quad e^A := I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \dots + \frac{1}{k!} A^k + \dots$$

(1220: notice this converges for any $A_{n \times n}$, since if ~~the n th power~~ $|a_{ij}| \leq M \quad \forall i, j$

$$\text{then } |\text{entry}_{ij}(A^2)| \leq nM^2$$

$$|\text{entry}_{ij}(A^3)| \leq n^2 M^3$$

$$|\text{entry}_{ij}(A^k)| \leq n^{k-1} M^k$$

so the ij entry series

converges absolutely

(dominated by the series for e^{Mn})

$$e^{tA} := I + tA + \frac{t^2}{2!} A^2 + \frac{t^3}{3!} A^3 + \dots + \frac{t^k}{k!} A^k + \dots$$

notice, for $X(t) := e^{tA}$

$$\begin{cases} X(0) = I \\ X'(t) = A + \frac{2t}{2!} A^2 + \frac{t^2}{2!} A^3 + \dots + \frac{t^{k-1}}{(k-1)!} A^k + \dots \\ \quad = A \left(I + tA + \frac{t^2}{2!} A^2 + \dots \right) \\ \quad = AX \end{cases}$$

(Maybe in 3220
you'll learn why
you can diff this
 ∞ series term by term)

Theorem

Since the power series def of e^{tA} solves $\begin{cases} X' = AX \\ X(0) = I \end{cases}$ (and sol'n is unique)
It must be that

$$e^{tA} = \Phi(t) \Phi^{-1}(0) = I + tA + \frac{t^2}{2!} A^2 + \dots + \frac{t^k}{k!} A^k + \dots$$

Example 1 cont'd (2270!!)

5

$$A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

$$S = \left[\vec{v}_1 \mid \vec{v}_2 \right] = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} = S^{-1}AS$$

so $A = S\Lambda S^{-1}$

$$A^2 = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda^2 S^{-1}$$

$$A^k = S\Lambda^k S^{-1}$$

← notice $\Lambda^k = \begin{bmatrix} (-2)^k & 0 \\ 0 & 5^k \end{bmatrix}$ is easy to compute!

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \dots + \frac{t^k}{k!}A^k + \dots$$

$$= I + tS\Lambda S^{-1} + \frac{t^2}{2!}S\Lambda^2 S^{-1} + \dots + \frac{t^k}{k!}S\Lambda^k S^{-1}$$

$$= S \left[I + t\Lambda + \frac{t^2}{2!}\Lambda^2 + \dots + \frac{t^k}{k!}\Lambda^k + \dots \right] S^{-1}$$

$$= S \begin{bmatrix} \sum_{k=0}^{\infty} \frac{(-2t)^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{(5t)^k}{k!} \end{bmatrix} S^{-1}$$

$$= \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{5t} \end{bmatrix} \frac{1}{7} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & -2e^{-2t} \\ 3e^{5t} & e^{5t} \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} e^{-2t} + 6e^{5t} & -2e^{-2t} + 2e^{5t} \\ -3e^{-2t} + 3e^{5t} & 6e^{-2t} + e^{5t} \end{bmatrix}$$

same as page 3!!!