Math 2280-1
Friday March 12

HW for 3/19 (1)

- Maple exploration §5.3
Understand set up, then do the project work in the paragraph at bottom of page 331; "For your personal seven-story ..." You don't need to hand in the warm-up problems.

5.4 (1, 7, 11, 29) Work 1, 7, 11 by hand, except for computer to draw #1 phase portrait. You may work 29 by hand, but I'd recommend Maple to help you find chains.

5.5 (1, 3, 11, 23, 36) Work by hand, except Maple for eigenvectors. Also have Maple check exp(At), and hand in.

§5.4 : 1^{st} order systems $\frac{d\vec{x}}{dt} = A\vec{x}$

when A is not diagonalizable

Recall (2270),
for any $A_{n\times n}$,

$$|A-\lambda I| = (-1)^n(\lambda-\lambda_1)^{k_1}(\lambda-\lambda_2)^{k_2}\cdots(\lambda-\lambda_m)^{k_m} \qquad \lambda_i \in \mathbb{C}, \quad \lambda_i \text{ distinct}, \quad 1 \le i \le m$$

$$k_1 + k_2 \cdots + k_m = n$$

$E_{\lambda_i} = \lambda_i$-eigenspace

- $\dim(E_{\lambda_i}) \le k_i$ always holds (geometric multiplicity $\le$ algebraic mult.)
- A is diagonalizable (has an $\mathbb{R}^n$ eigenbasis) iff each $\dim(E_{\lambda_i}) = k_i$

and in this (diagonalizable) case, we can find the general soltn to $\frac{d\vec{x}}{dt} = A\vec{x}$ as linear combos of $\{e^{\lambda_1 t}\vec{v}_1, \ldots e^{\lambda_m t}\vec{v}_n\}$.

($\exists$ analog for complex evals & evects)

example 1 p 333

$$\vec{x}' = \begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix}\vec{x}$$

$$|A-\lambda I| = -(\lambda-5)(\lambda-3)^2$$

$\lambda = 5$

$$\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda = 3$:

$$\left[\begin{array}{ccc|c} 6 & 4 & 0 & 0 \\ -6 & -4 & 0 & 0 \\ 6 & 4 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t$
$x_2 = s$
$x_1 = -\frac{2}{3}s$

$$\vec{x} = s\begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\{\vec{v}_1, \vec{v}_2\} = \left\{\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$$\vec{x}_H(t) = c_1 e^{5t}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + c_3 e^{3t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

all is well!
(even though $\lambda = 3$ had algebraic mult = 2).

But, how about

$$\vec{x}' = \underbrace{\begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -9 & 9 \end{bmatrix}}_{B} \vec{x}$$

$$|B-\lambda I| = \begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & -3-\lambda & 4 \\ 0 & -9 & 9-\lambda \end{vmatrix} = (5-\lambda)\left[\lambda^2 - 6\lambda \overbrace{-27+36}^{+9}\right]$$
$$= (5-\lambda)(\lambda-3)^2 \qquad \text{same as example 1.}$$

$\lambda = 5$ $\qquad$ $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda = 3$ $\qquad$ $\vec{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ $\qquad$ (Maple)

$\lambda = 3$ eigenspace is "defective"

$$\left\{ e^{5t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ e^{3t}\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix},\ ?? \right\}$$

discussion:

Let $A\vec{v} = \lambda\vec{v}$, but λ-eigenspace "defective".

$e^{\lambda t}\vec{v}$ solves $\dfrac{d\vec{x}}{dt} = A\vec{v}$

does $\vec{z}(t) = te^{\lambda t}\vec{v}$?

$$\frac{d\vec{z}}{dt} = e^{\lambda t}\vec{v}\left[1 + \lambda t\right]$$
$$A\vec{z} = te^{\lambda t}A\vec{v} = te^{\lambda t}\lambda\vec{v}$$

$$\frac{d\vec{z}}{dt} - A\vec{z} = e^{\lambda t}\vec{v} \qquad \text{so } \vec{z}(t) \text{ fails.}$$

how about

$$\vec{z}(t) = e^{\lambda t}(t\vec{v} + \vec{w}) \ ?$$
$$\vec{z}'(t) = e^{\lambda t}\left[\lambda t\vec{v} + \lambda\vec{w} + \vec{v}\right]$$
$$A\vec{z} = e^{\lambda t}\left[t\lambda\vec{v} + A\vec{w}\ \right]$$

success iff $\boxed{A\vec{w} = \lambda\vec{w} + \vec{v}}$

$\boxed{(A-\lambda I)\vec{w} = \vec{v}}$

in our example: $\left(\lambda = 3,\ \vec{v} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}\right)$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & -6 & 4 & 2 \\ 0 & -9 & 6 & 3 \end{array}\right]$$

$$\begin{array}{l} \\ R_2/-2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 3 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$w_3 = t$
$w_2 = -\frac{1}{3} + \frac{2}{3}t$
$w_1 = 0$

$$\vec{w} = \begin{bmatrix} 0 \\ -1/3 \\ 0 \end{bmatrix} + \underbrace{t\begin{bmatrix} 0 \\ 2/3 \\ 1 \end{bmatrix}}_{\text{mults of } \vec{v}}$$

can take $\vec{w} = \begin{bmatrix} 0 \\ -1/3 \\ 0 \end{bmatrix}$

get soln

$$?? = \vec{z}(t) = e^{\lambda t}\left\{ t\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/3 \\ 0 \end{bmatrix} \right\}$$

this gives a 3rd lin ind sol'n!!

<u>Theorems</u> : Let $A_{n\times n}$ as on page 1, with

$$|A-\lambda I| = (-1)^n (\lambda-\lambda_1)^{k_1}(\lambda-\lambda_2)^{k_2}\cdots(\lambda-\lambda_m)^{k_m}$$

$k_1+\cdots+k_m=n$
$\lambda_i \in \mathbb{C} \quad 1\le i\le m$
λ_i distinct

(1) The <u>generalized</u> λ_i-eigenspace is defined to be

$$\ker\,(A-\lambda_i I)^{k_i}$$

it <u>is</u> always k_i dimensional (in $\mathbb{C}^n$)

(2) By amalgamating bases of the generalized eigenspaces you get a basis for $\mathbb{C}^n$

(3) The generalized λ_i-eigenbases can be chosen to be made of one or more "chains" $\{\vec{u}_1, \vec{u}_2, \ldots \vec{u}_k\}$ where for $\lambda=\lambda_i$:

$$\begin{cases} (A-\lambda I)\vec{u}_1 = \vec{0} \\ (A-\lambda I)\vec{u}_2 = \vec{u}_1 \\ \vdots \\ (A-\lambda I)\vec{u}_k = \vec{u}_{k-1} \end{cases}$$

($\vec{u}_1$ <u>is</u> an eigenvector)

(~~ex~~ on page 2, $(\vec{v},\vec{w})$ was a chain of length 2)

(4) Each chain $\{\vec{u}_1,\vec{u}_2,\ldots\vec{u}_k\}$ yields k lin. ind. sol'ns to $\frac{d\vec{x}}{dt}=A\vec{x}$:

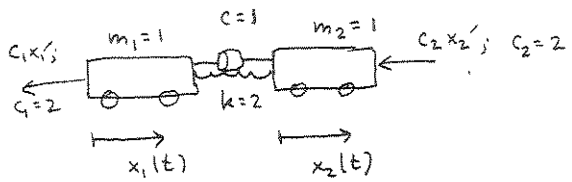
$$\vec{x}_1(t) = \vec{u}_1 e^{\lambda t}$$
$$\vec{x}_2(t) = (\vec{u}_1 t + \vec{u}_2)e^{\lambda t}$$
$$\vec{x}_3(t) = \left(\vec{u}_1 \tfrac{1}{2}t^2 + \vec{u}_2 t + \vec{u}_3\right)e^{\lambda t}$$
$$\vdots$$
$$\vec{x}_{k}(t) = \left(\vec{u}_1 \tfrac{1}{k!}t^k + \cdots + \vec{u}_{k-1}t + \vec{u}_k\right)e^{\lambda t}$$

$\leftarrow$ notice $\quad \vec{x}_j{}' = \lambda\vec{x}_j + \vec{x}_{j-1}$
$A\vec{x}_j = \lambda\vec{x}_j + \vec{x}_{j-1}$ too!

amalgamating these solutions yields a basis of sol'ns to $\frac{d\vec{x}}{dt}=A\vec{x}$!!!

<u>proof</u> we will discuss aspects of the general proof during following lectures
(might)

Example 6 p. 343



$$m_1 x_1'' = +k(x_2 - x_1) - c_1 x_1' + c(x_2' - x_1')$$
$$m_2 x_2'' = -k(x_2 - x_1) - c_2 x_2' - c(x_2' - x_1')$$

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

cannot use §5.3 (no damping.)
Must convert to 1st order system

$x_1 = x_1$
$x_2 = x_2$
$x_3 = x_1'$
$x_4 = x_2'$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Maple to the rescue! (See next page).

$$\begin{bmatrix} x_1 \\ x_2 \\ x_1' \\ x_2' \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + e^{-2t}\left[c_2 \begin{bmatrix} 1 \\ 1 \\ -2 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix} + c_4 \overbrace{\left[t \begin{bmatrix} -1 \\ 1 \\ 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right]}^{t\vec{u}_1 + \vec{u}_2} \right]$$

explain.

nice physically; the sum of MAPLE's 2 $\lambda = -2$ eigenvects, times -2.

the difference of Maple's 2 $\lambda=-2$ eigenvects, $\times 2$. also nice physical interpretation

```
> with(LinearAlgebra) :
> A := Matrix(4, 4, [0, 0, 1, 0, 0, 0, 0, 1,
        -2, 2, -3, 1, 2, -2, 1, -3]);
```

$$A := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 2 & -3 & 1 \\ 2 & -2 & 1 & -3 \end{bmatrix} \quad (1)$$

```
> Eigenvectors(A);
```

$$\begin{bmatrix} 0 \\ -2 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (2)$$

$$> Iden := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} :$$

```
> B := A + 2·Iden; # eigenvalue is -2
```

$$B := \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -2 & 2 & -1 & 1 \\ 2 & -2 & 1 & -1 \end{bmatrix} \quad (3)$$

```
> NullSpace(B^3); #generalized eigenspace should be 3-dimensional
```

$$\left\{ \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \quad (4)$$

```
> NullSpace(B^2); #I need a chain of length 2 - there won't be one of length 3
```

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad (5)$$

```
> u2 := Vector([0, 0, -1, 1]); #for chain, don't pick eigenvector - ths generalized
                                #eigenvector has a nice meaning for the train system
  u1 := B.u2; #note that u2 is in the regular eigenspace.
```

$$u2 := \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$u1 := \begin{bmatrix} -1 \\ 1 \\ 2 \\ -2 \end{bmatrix} \quad (6)$$