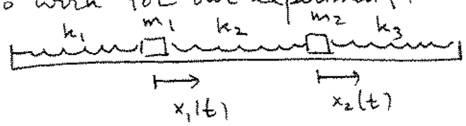
Math 2280-1
Wednesday March 10

Solve the particular coupled mass-spring system

$$M\vec{x}'' = K\vec{x} \quad \rightsquigarrow \quad \vec{x}'' = A\vec{x} \qquad A = M^{-1}K$$

from yesterday's notes.
Let's put that example into a slightly more general context which will also work for our experiment:



$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1)$$
$$m_2 x_2'' = -k_2(x_2 - x_1) - k_3 x_2$$

$$m_1 x_1'' = -(k_1+k_2)x_1 + k_2 x_2$$
$$m_2 x_2'' = k_2 x_1 \quad - (k_2+k_3)x_2$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -k_1-k_2 & k_2 \\ k_1 & -k_2-k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$M\vec{x}'' = K\vec{x}$, also for n masses coupled with $n+1$ springs

$$\boxed{\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_1}{m_2} & -\frac{k_2+k_3}{m_2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

$\vec{x}'' = A\vec{x}$; $A = M^{-1}K$.

soltn space 4-dim'l

soltn space $2n$-dim'l.

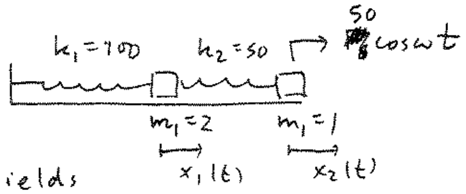
seek basis fcns of form
$\cos\omega t\,\vec{v}$, $\sin\omega t\,\vec{v}$; plugging into $\vec{x}'' = A\vec{x}$
yields $-(\cancel{\cos\omega t})\omega^2\vec{v} = \cancel{\cos\omega t}\,A\vec{v}$
or $-(\cancel{\sin\omega t})\omega^2\vec{v} = \cancel{\sin\omega t}\,A\vec{v}$
so $\vec{v}$ must be an eigenvector of A
with $A\vec{v} = -\omega^2\vec{v}$

(so, hopefully A has n real ind eigenvectors, each with negative eigenvalues, so that $\omega = \sqrt{-\lambda}$ makes sense as an angular freq.
$\sim$ should yield $2n$ ind. sol'ns!

now work Tues. example:
$k_1 = 100$, $k_2 = 50$, $m_1 = 2$
$k_3 = 0$ $\quad m_2 = 1$

Now force on system @ angular frequency ω, pulling/pushing on the second mass.

This is example 3 p. 327



yields

$$\vec{x}'' = \begin{bmatrix} -75 & 25 \\ 50 & -50 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 50 \end{bmatrix} \cos \omega t$$

try $\vec{x}_p = \vec{c} \cos \omega t$ unknown $\vec{c}$

yields

$$-\omega^2 \vec{c} \cancel{\cos \omega t} = A \vec{c} \cancel{\cos \omega t} + \begin{bmatrix} 0 \\ 50 \end{bmatrix} \cancel{\cos \omega t}$$

$$-\omega^2 I \vec{c} = A \vec{c} + \begin{bmatrix} 0 \\ 50 \end{bmatrix}$$

$$\left[A + I\omega^2\right] \vec{c} = \begin{bmatrix} 0 \\ -50 \end{bmatrix}$$

$$\vec{c} = \left[A + I\omega^2\right]^{-1} \begin{bmatrix} 0 \\ -50 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -75+\omega^2 & 25 \\ 50 & -50+\omega^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -50 \end{bmatrix}$$

$$= \frac{1}{\underbrace{(\omega^2-75)(\omega^2-50) - 50 \cdot 25}_{\omega^4 - 125\omega^2 - 2500}} \begin{bmatrix} -50+\omega^2 & -25 \\ -50 & -75+\omega^2 \end{bmatrix} \begin{bmatrix} 0 \\ -50 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{(\omega^2-25)(\omega^2-100)} \begin{bmatrix} 1250 \\ 50(75-\omega^2) \end{bmatrix}$$

$\vec{x}_p = \cos \omega t \, \vec{c}$

notice that as ω approaches 5 or 10

$\vec{x}_p$ amplitudes $\to \infty$!

In a slightly damped system (where $\vec{x}_H \to 0$), this would mean practical resonance – see Maple!

$$\vec{x}'' = A\vec{x} + \vec{F}_0 \cos \omega t$$

$$A = M^{-1}K$$
$$\vec{F}_0 = M^{-1}\vec{C}$$

original force amplitude vector

$\vec{x}_p = \cos \omega t \, \vec{c}$

$$-\omega^2 \cancel{\cos \omega t} \, \vec{c} = \cancel{\cos \omega t} A\vec{c} + \vec{F}_0 \cancel{\cos \omega t}$$

$$-\omega^2 I \vec{c} = A\vec{c} + \vec{F}_0$$

$$-\vec{F}_0 = (A + \omega^2 I)\vec{c}$$

$$\boxed{\vec{c} = -(A + \omega^2 I)^{-1} \vec{F}_0}$$

works unless $A + \omega^2 I$ is singular, i.e. unless $\omega^2 = -\lambda$ for some eigenvalue of A, i.e. unless we force at a natural frequency and induce resonance!

MATH 2280-1
Mass-Spring systems
March 10, 2010

This handout covers section 5.3 and might help for the Earthquake exploration you're doing at the end of that section, on pages 330-332. You are mostly on your own for this project, but here is a small example of a spring system worked out on Maple, so that you can get an idea about useful commands to use. This is the example we set up yesterday, but we didn't get very far. Consult yesterday's notes as we go through this example.

This is example 1 on page 321 of Edwards-Penney, which we just worked by hand. Initially it is an unforced system with two masses and two springs, as you can see from the description in EP. We can write the system as **Mx''=Kx**, where **M** is the ``mass matrix", **K** is the ``spring matrix", and **x** is the displacement vector. Following the book's notation, we enter

```
> with(LinearAlgebra) : with(plots) : with(DEtools) :
    #useful command libraries
> M := Matrix([[2, 0], [0, 1]]);
  K := Matrix([[-150, 50], [50, -50]]);
  A := M^(-1).K; #remember, the period . works for matrix multiplication
    #A is the matrix in the system x''=Ax
```

$$M := \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K := \begin{bmatrix} -150 & 50 \\ 50 & -50 \end{bmatrix}$$

$$A := \begin{bmatrix} -75 & 25 \\ 50 & -50 \end{bmatrix} \qquad (1)$$

As we've realized, the eigenvectors of A determine fundamental modes of the spring system, and the corresponding negative eigenvalues are the (opposites) of the squares of the corresponding angular frequencies. Let's check our linear algebra:

```
> Eigenvectors(A);
```

$$\begin{bmatrix} -100 \\ -25 \end{bmatrix}, \begin{bmatrix} -1 & \frac{1}{2} \\ 1 & 1 \end{bmatrix} \qquad (2)$$

Therefore, as we discussed, the natural frequencies of this system are the 10 and 5, and the two fundamental modes correspond to the masses moving in opposite directions (with equal amplitudes and angular frequency 10) and in parallel directions (with amplitude ratio of two and angular frequency 5).

Here's the Maple version of our forced oscillation work:

```
> F0 := M^(-1).Vector([0, 50]);
    #the F0 in the normalized equation (29), page 326.
  Iden := [[1, 0], [0, 1]];
```

$$F0 := \begin{bmatrix} 0 \\ 50 \end{bmatrix}$$

$$Iden := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (3)$$

```
> AL := omega -> A + ω^2·Iden; #the matrix function multiplying c in equation (32),
    #on the left. The "star" symbol · is for scalar time matrix
  AL(omega);
  c := omega -> -(AL(omega))^(-1).F0; #not sure what that output is saying!
  c(omega); #but it seems to work -see equation (35), page 327.
```

$$AL := \omega \rightarrow A + \omega^2\ Iden$$

$$\begin{bmatrix} -75 + \omega^2 & 25 \\ 50 & -50 + \omega^2 \end{bmatrix}$$

$$c := \omega \rightarrow -Typesetting\text{:-}delayDotProduct\left(\frac{1}{AL(\omega)}, F0\right)$$

$$\begin{bmatrix} \dfrac{1250}{2500 - 125\,\omega^2 + \omega^4} \\ -\dfrac{50\left(-75 + \omega^2\right)}{2500 - 125\,\omega^2 + \omega^4} \end{bmatrix} \qquad (4)$$

Thus if we force the system as indicated, and if is not one of the two natural frequencies, we get a particular solution

$$\cos(\omega t)\begin{bmatrix} \dfrac{1250}{2500-125\,\omega^2+\omega^4} \\ -\dfrac{50\left(-75+\omega^2\right)}{2500-125\,\omega^2+\omega^4} \end{bmatrix}.$$

If we assume that our actual problem has a small amount of damping, then we expect that this particular solution is very close to the steady periodic solution to the damped problem. See the discussion on page 327. We can study practical resonance phenomena for these slightly damped problems by plotting the maximum amplitude of the steady state solutions to the undamped problems. That would be the maximum absolute value of c1 and c2 above. Use the Maple command ``norm" to measure this maximum amplitude:

```
> norm(c(omega));
```

$$\max\left(\frac{1250}{\left|2500-125\,\omega^2+\omega^4\right|},\ 50\left|\frac{-75+\omega^2}{2500-125\,\omega^2+\omega^4}\right|\right) \tag{5}$$

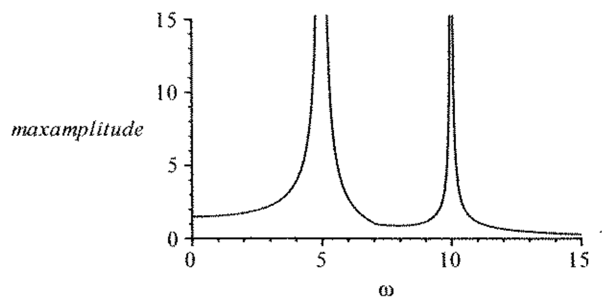
Another way to measure the size of c(omega) is to take its Euclidean magnitude, which is the command

```
> norm(c(omega),2);
```

$$50\sqrt{\frac{625}{\left|2500-125\,\omega^2+\omega^4\right|^2}+\left|\frac{-75+\omega^2}{2500-125\,\omega^2+\omega^4}\right|^2} \tag{6}$$

(You will use the first command in the Earthquake project, which perhaps makes the most sense since it will be measuring the maximum amplitude that any floor oscillates.) The following picture illustrates that the maximum amplitude of the particular solution blows up when omega is near the two natural angular frequences. Thus, in the slightly damped problem, one would experience practical resonance in the steady periodic solution.

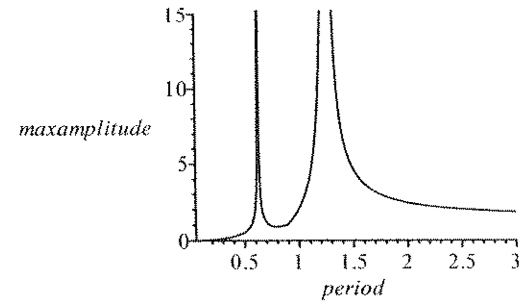
```
> plot(norm(c(omega)),omega=0..15,maxamplitude=0..15,
        numpoints=200,color=`black`);
```



This is qualitatively the picture on page 327, figure 5.3.10, although they plotted the Euclidean magnitude of c(omega) rather than the maximum amplitude. Notice how we get Maple to label the axes as desired

We can get a plot of resonance as a function of period by recalling that 2*Pi/T=omega:

```
> plot(norm(c(2*Pi/period)),period=0.1..3,maxamplitude=0..15,
  numpoints=200,color=`black`);
```



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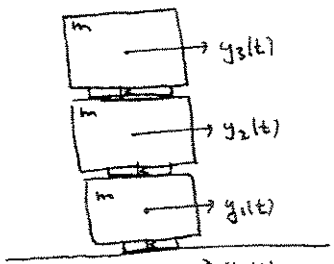
Earthquake Project Comments

(A careful explanation of the book's claim that even though the building is only being shaken at the base, in a frame of reference moving with ground, each floor experiences an equal "inertial force".)

consider a 3 story building (in project is 7 stories) modeled as a mass-spring model, with each story having mass m, and floor junctions modeled as "springs" with constants k (with respect to transverse as opposed to parallel motion)



$\rightarrow$ transverse (shaking) wave $E\cos\omega t$

$y_0(t) = E\cos\omega t$ so

$$y_0'' = -E\omega^2\cos\omega t$$
$$m y_1'' = -k(y_1-y_0) + k(y_1-y_2)$$
$$m y_2'' = -k(y_2-y_1) - k(y_2-y_3)$$
$$m y_3'' = -k(y_3-y_2)$$

reduction of fns

let

$$x_0 = y_0 - E\cos\omega t \equiv 0$$
$$x_1 = y_1 - E\cos\omega t$$
$$x_2 = y_2 - E\cos\omega t$$
$$x_3 = y_3 - E\cos\omega t$$

$$x_0 \equiv 0$$
$$m x_1'' = m y_1'' + mE\omega^2\cos\omega t$$
$$m x_2'' = m y_2'' + mE\omega^2\cos\omega t$$
$$m x_3'' = m y_3'' + mE\omega^2\cos\omega t$$

so

$$m x_1'' = -k(x_1-0) - k(x_1-x_2) + mE\omega^2\cos\omega t$$
$$m x_2'' = -k(x_2-x_1) - k(x_2-x_3) + mE\omega^2\cos\omega t$$
$$m x_3'' = -k(x_3-x_2) \qquad + mE\omega^2\cos\omega t$$

$$\boxed{\begin{bmatrix} x_1'' \\ x_2'' \\ x_3'' \end{bmatrix} = \begin{bmatrix} -2k/m & k/m & 0 \\ k/m & -2k/m & k/m \\ 0 & k/m & -k/m \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + E\omega^2\cos\omega t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

so the x's are the displacements from equilibrium as measured by an observer on the ground

Notice $y_i - y_j = x_i - x_j$

it's as if all floors are being forced – book calls this inertial forcing due to frame of reference