Math 2280-1
Monday Jan. 25

§2.1 Improved population models

Let $P(t)$ = population (e.g. # people, # fish, etc.)
$B(t)$ = birth rate (e.g. people/year)
$D(t)$ = death rate (e.g. people/year)

$$\beta(t) = \frac{B(t)}{P(t)} \qquad \text{(people/year)/people} \qquad \text{e.g. } .1 \text{ person/year per person.}$$

fertility rate

$$\delta(t) = \frac{D(t)}{P(t)} \qquad \text{morbidity rate}$$

Then

$$\frac{dP}{dt} = B(t) - D(t)$$

$$\boxed{\frac{\left(\frac{dP}{dt}\right)}{P} = \beta(t) - \delta(t)}$$

model 1 : $\beta(t) \equiv \beta$ const, $\delta(t) \equiv \delta$ const $\Rightarrow \frac{dP}{dt} = (\beta - \delta)P = kP$

either exponential growth or decay, depending on sign of k ... we know this DE!

model 2 : $\beta = \beta_0 - \beta_1 P$, $\delta = \delta_0 + \delta_1 P$ (β_0, β_1 const)

$\beta_1 > 0$ "sophisticated" population
$\delta_1 > 0$ ~ finite resources, disease, fighting

So

$$\begin{aligned}\frac{dP}{dt} &= (\beta - \delta)P \\ &= [(\beta_0 - \beta_1 P) - (\delta_0 + \delta_1 P)]P \\ &= [(\beta_0 - \delta_0) - (\beta_1 + \delta_1)P]P\end{aligned}$$

$$\boxed{\begin{aligned}\frac{dP}{dt} &= kP(M - P) \\ \frac{dP}{dt} &= aP - bP^2\end{aligned}}$$

$k = \beta_1 + \delta_1$, $M = \frac{\beta_0 - \delta_0}{\beta_1 + \delta_1}$

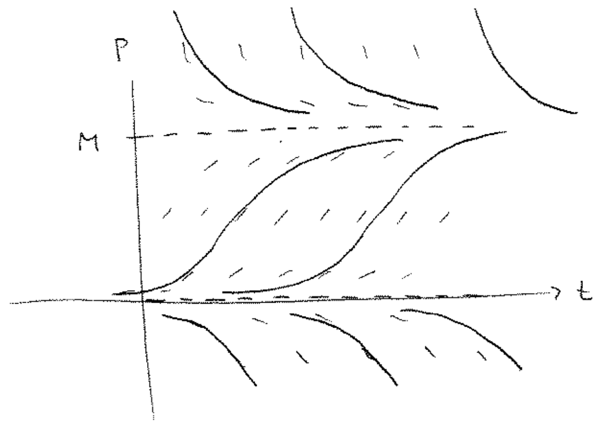
$a = kM$, $b = k$ $\left(\begin{array}{l} k = b \\ M = a/b \end{array}\right)$

Logistic eqtn

(There is a model 3 we'll discuss Monday for which $\beta = \beta_0 + \beta_1 P$ ("alligators") which can lead to the "doomsday-extinction" de, $\frac{dP}{dt} = -aP + bP^2$.)

What does the slope field predict for sol'ns to

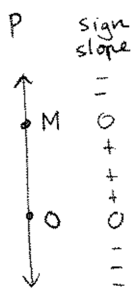
$$\text{IVP} \begin{cases} \dfrac{dP}{dt} = kP(M-P) & (k, M > 0) \\ P(0) = P_0 \end{cases}$$



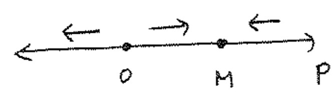
slope field picture

notice isoclines are horizontal lines in the (t, P) plane:

$P = 0 \Rightarrow m = 0$
$P = M \Rightarrow m = 0$



leads to phase portrait:



$P = 0$ and $P = M$ are called <u>equilibrium solutions</u> (because they are constant in time)

- It appears that <u>any</u> IVP sol'n $P(t)$ with $P(0) = P_0 > 0$ will converge to $P = M$.

$\boxed{M := \text{carrying capacity.}}$

$\boxed{\text{Solve the logistic IVP by separating variables}}$ (Use back of page if necessary!)
(Then analyze solution to verify slope field & phase portrait predictions)

$$\text{Sol: } P(t) = \frac{M}{\left(\frac{M-P_0}{P_0}\right)e^{-Mkt} + 1}$$

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Examples 4,5 from %2.1 of the text, pages 83-84.

The Belgian demographer P.F. Verhulst introduced the logistic model around 1840, as a tool for studying human population growth. Our text demonstrates its superiority to the simple exponential growth model, and also illustrates why mathematical modelers must always exercise care, by comparing the two models to actual U.S. population data. here are actual U.S. populations from 1800-1990, see e.g. the table on page 84:

```
> restart:  #clear Maple memory
> pops:= [[1800,5.3],[1810,7.2],[1820,9.6],[1830,12.9],
          [1840,17.1],[1850,23.2],[1860,31.4],[1870,38.6],
          [1880,50.2],[1890,63.0],[1900,76.2],[1910,92.2],
          [1920,106.0],[1930,123.2],[1940,132.2],[1950,151.3],
          [1960,179.3],[1970,203.3],[1980,225.6],[1990,248.7]]:
```

Unlike Verhulst, the book uses data from 1800, 1850 and 1900 to get constants in our two models. We let t=0 correspond to 1800.

Exponential Model: For the exponential growth model $P(t) = P_0\, e^{r t}$ we use the 1800 and 1900 data to get values for P_0 and r:

```
> P0:=5.308;
  solve(P0*exp(r*100)=76.212,r);
```

$$P0 := 5.308$$

$$0.02664303814 \quad (1)$$

```
> P1:=t->5.308*exp(.02664*t);  #exponential model
```

$$P1 := t \rightarrow 5.308\, e^{0.02664\, t} \quad (2)$$

Logistic Model: We get P_0 from 1800, and use the 1850 and 1900 data to find k and M:

```
> P2:=t->M*P0/(P0+(M-P0)*exp(-M*k*t));
     #logisitic function, with our P0
```

$$P2 := t \rightarrow \frac{M\, P0}{P0 + (M - P0)\, e^{-M k t}} \quad (3)$$

```
> solve({P2(50)=23.192, P2(100)=76.212},{M,k});
```

$$\{M = 188.1208275,\ k = 0.0001677157274\} \quad (4)$$

```
> M:=188.1208275;
  k:=.1677157274e-3;
  P2(t);  #should be our logistic model function,
     #equation (11) page 82.
```

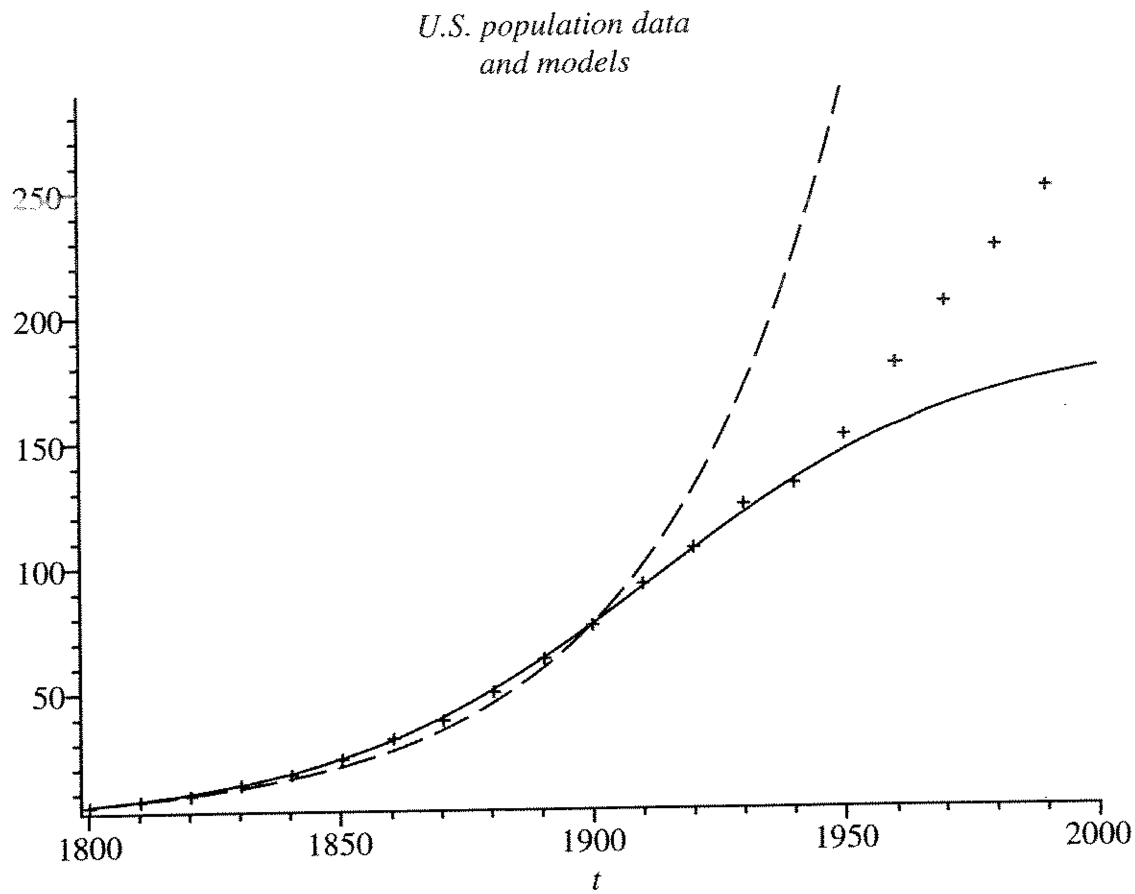
$$M := 188.1208275$$

$$k := 0.0001677157274$$

$$\frac{998.5453524}{5.308 + 182.8128275\, e^{-0.03155082142\, t}} \quad (5)$$

Now compare the two models with the real data, and discuss:

```
> with(plots):
  plot1:=plot(P1(t-1800),t=1800..1950,color=black, linestyle=3):
    #this linestyle gives dashes for the exponential curve
  plot2:=plot(P2(t-1800),t=1800..2000,color=black):
  plot3:=pointplot(pops,symbol=cross):
  display({plot1,plot2,plot3},title=`U.S. population data
  and models`);
```



The exponential model takes no account of the fact that the U.S. has only finite resources. Any ideas on why the logistic model begins to fail (with our parameters) around 1950?