Math 2280-1
Jan 19, 2010

On Friday you discussed the existence-uniqueness theorem for solutions to

$$\text{IVP} \begin{cases} \frac{dy}{dx} = f(x,y) \\ y(a) = b \end{cases}$$

Here are some more interesting examples in §1.4

<u>Exercise 1</u> Consider the differential equation $\frac{dy}{dx} = \frac{4-2x}{3y^2-5}$ (p. 34)

1a) Find the separation of variables solutions. What assumption do you make?

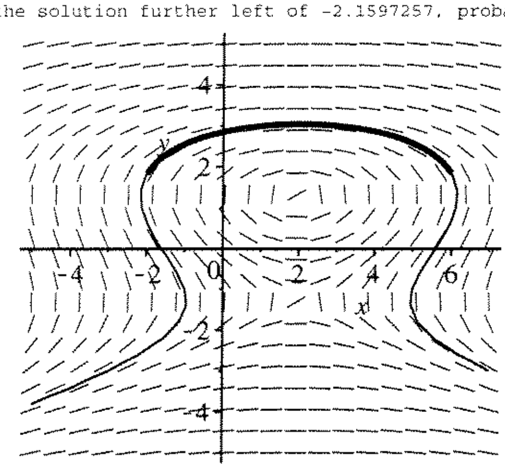
1b) Discuss ways in which the implicit solutions can be made explicit.
Do all solns exhibit some symmetry?

1c) Relate discussion to MAPLE output

```
> with(plots) : with(DEtools) :
  deqtn := diff(y(x), x) = (4 - 2·x)/(3·y(x)^2 - 5);
  part1 := DEplot(deqtn, y(x), x=-5..7, y=-5..5, {[y(1) = 3]}, arrows = line, color = black, linecolor = blue) :
  part2 := implicitplot(y^3 - 5·y = 4·x - x^2 + 9, x=-5..7, y=-5..5, color = black) :
  display({part1, part2});
```

$$deqtn := \frac{d}{dx}y(x) = \frac{4-2x}{3y(x)^2-5}$$

```
Warning, plot may be incomplete, the following errors(s) were issued:
   cannot evaluate the solution further right of 6.1597255, probably a singularity
   cannot evaluate the solution further left of -2.1597257, probably a singularity
```




```
>
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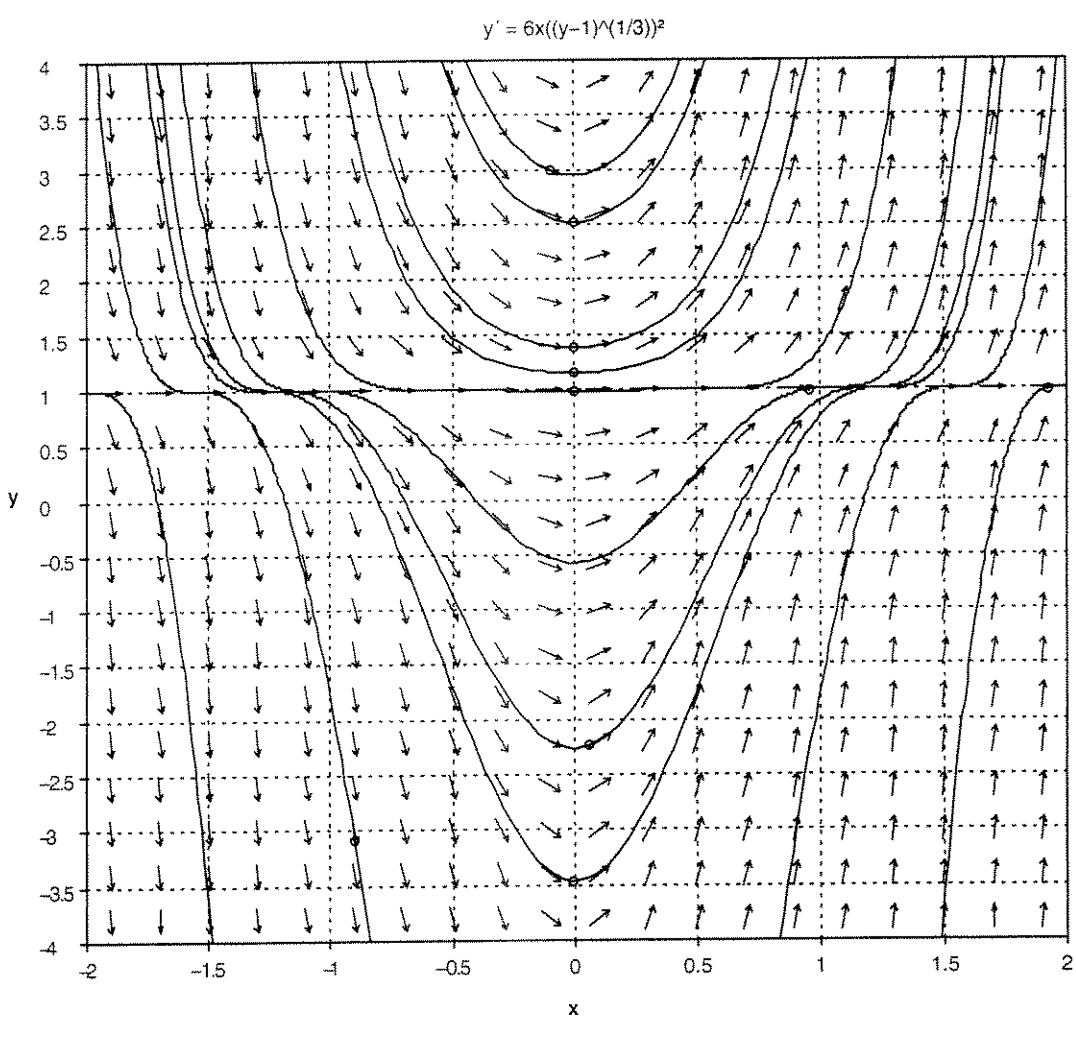
Exercise 2 (Example 4 p.36)

Find all solutions to the DE

$$\frac{dy}{dx} = 6x(y-1)^{2/3}$$

Notice the singular solutions!!

dfield output



In §1.4 text & hw there are exponential growth/decay + Newton's law of cooling examples. These should be review for you, especially after last week's examples.

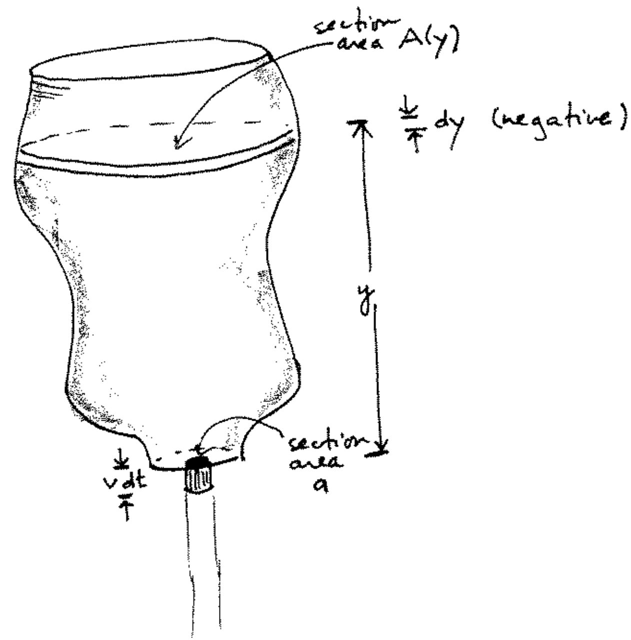
Here's a model that might be new to you:

Toricelli's Law for draining tanks

Let a tank be draining water. Let $y(t)$ be the water height above the drain. Then the speed $v$ with which the water leaves the tank is given by

$$\boxed{v = \sqrt{2gy}}$$

$g$ = acceleration of gravity.




derivation neglecting friction yields a conservative system for which

$$KE + PE = \text{const}$$

in a small time interval $dt$ there is a (negative) change of water mass in the tank, $dM$.

An equal mass has drained from the drain at speed $v$.

By conservation of energy,

loss in PE = gain in KE

$$\boxed{\cancel{|dM|}\, g\, y = \tfrac{1}{2}\cancel{|dM|}\, v^2}$$

thus $\boxed{v = \sqrt{2gy}}$

This yields a separable differential equation for $y(t)$

$dM = \rho A(y) dy$ ← $\rho$ = water density mass/vol. (1 gm/cc)

$dM = \rho a(-v\,dt)$ ←

measure $dM$ at top of water level

measure $dM$ ~~[illegible]~~ at the drain

thus $A(y)dy = -av\,dt$

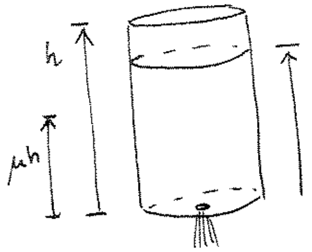
$A(y)dy = -a\sqrt{2gy}\, dt$, by Torricelli

$$\boxed{A(y)\frac{dy}{dt} = -k\sqrt{y}}$$

$k = a\sqrt{2g}$

Experiment!!

**Exercise 3** Cylindrical cistern: $A(y) = A$ const.




$$\frac{dy}{dt} = -k y^{1/2} \quad \text{(different } k\text{).}$$

Let $T_\mu$ be the amount of time it takes the cistern to empty from full height h, to height $\mu h$ $(0 \le \mu \le 1)$
Show the time it takes to empty the tank is given by $\boxed{T_0 = \frac{T_\mu}{1-\mu^{1/2}}}$

Nalgene bottle experiment:
I marked off the bottle so that we can use $\mu = .5$
So if we time how long it takes to empty half the height, and call it $T_{1/2}$,
then the total time estimate will be

$$T_0 = \frac{1}{1-\sqrt{.5}} T_{.5} \approx (3.41) T_{.5}$$

Experiment

$T_{1/2} =$

$(3.41) T_{1/2} =$

$T_0 =$