

Math 2280–1

MAPLE

1.3–1.4: separable DE's, slope fields, existence and uniqueness.

Friday January 15, 2010

We'll work with this Maple document AND your class notes from Wednesday, so make sure to bring them along.

Homework for Friday January 22: (Hand in underlined problems.)

Maple Introduction for Math 2270, linked from our home page. The purpose of that document is to introduce you to Maple, its help features, some useful commands, and its ability to create documents like these class notes. If you were in my section of 2270 this past fall, I hope you're already comfortable with Maple, and you've been through that introduction once already. But if you have not used Maple before or want a refresher, have another look. The examples in that document are for linear algebra; in this document are differential equations examples. To open a window with Maple 13 on the Math Department Unix system, open a terminal window and after the prompt ">", fill in the line so that it looks like:

...>xmmapleV13 &

and then type the <return> (= <enter>) key. In a few moments a new window running Maple 13 should appear. There is nothing to hand in from the Maple introduction.

Book homework:

1.3: 3, 6, 10, 11, 12, 13, 18, 21, 23, 29. In problem 6, consider the actual problem to be **6a**. Then also do

6b) verify $y(x) = x + C e^{-x}$ solves the DE, and convince yourself the curves you sketch on the slope field are consistent with this formula.

6c) Use dfield to draw the indicated solution graphs. (For 6a you may use dfield to draw the solution field and then add the solution curves by hand.)

In problem 29, consider the book problem to be **29a**. Then also do

29b) Get dfield to illustrate several of the non-unique solutions you verify exist in 29a.

Recall you can google the applet "dfield", but also that if you're using Math Department system the on-line applet won't print hard copies. We've made a copy of the applet you can call from a terminal window which will print hard copies. This copy is also called dfield.

1.4: 9, 12, 19, 22, 35, 41, 43, 46, 54, 66.

1.5: 1, 7, 13, 34, 36, 38, 41.

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I created today's notes using the software package Maple, to show you the kind of documents it is possible to create with Maple. (I used "document mode." Although this software has been produced by a private company called *MapleSoft* since 1988, it originated with Canadian government support at the University of Waterloo; the originators used the leaf on the Canadian flag to motivate the name "Maple."

You will be using Maple in Math 2280, in order to solve problems and do computer projects. You should think of this software as analogous to the Microsoft Word as a tool for writing papers – except Maple lets you create documents which combine text and mathematics. Alternately, you may use Maple to generate mathematical output, which you can then export to other documents. (That is closer to how engineers use Matlab, which is more efficient than Maple at doing extremely large computations, but lacks Maple's ability to create text documents and to easily do symbolic mathematical work.)

I will use Maple 13 in Math 2280. There are unfortunate backwards compatibility problems, which means that you won't be able to open a Maple 13 document in any of the previous versions. (You can try saving in Maple 8 format, from Maple 13, by choosing to save in "classic" .mws format. But even then I've encountered problems trying to open the result from a Maple with number less than 13. It might help if you remove all math output before saving.) If you like to work on the Marriott system (which has Maple 8 and Maple 11), or at Engineering (which I think has Maple 8 and Maple 12), then you may want to use Maple 11, 12, or 8 in the Math Department. You can open any of these using the terminal window command at the start of this document, with the appropriate number replacing the "13". The University bookstore sells a student version of Maple 13 (about \$130), which you may wish to purchase at some point, although it is not at all necessary to do so for Math 2280. Alternately, you can download Maple 13 for about \$100, directly from the MapleSoft website.

When you open any of these Maple versions you can find a "new user's tour" in a "help" window, and I recommend you take this tour to get an overview of Maple's capabilities. From then on, you will learn to use Maple the same way you learned to use Microsoft Word, i.e. by using it, using the help features, and asking friends, lab assistants, TAs and teachers when you get stuck.

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We will use today's notes to continue our discussion in Wednesday's notes of slope fields and of the existence–uniqueness theorem, section 1.3. We reminded ourselves on Wednesday why the separation of variables "differential magic" gives correct (implicitly defined) solutions to separable first order DEs, section 1.4. Here's a version of that discussion written in Maple – so you can see that Maple is able to display math equations, probably more nicely than MSWord, but not as nicely as "latex", which is the typesetting program mathematicians use to write their research papers. (The way you get equations to copy out of Math fields into text fields is to write a pseudo command in a math field which generates the output you want; then copy and paste that output into a text field.)

Separable Differential Equations

A first order differential equation

$$\frac{dy}{dx} = h(x, y)$$

is called **separable** iff $h(x,y)$ is a product of a function of x times a function of y ,

$$\frac{dy}{dx} = g(x) \phi(y) .$$

This is equivalent to the DE

$$\frac{dy}{dx} = \frac{g(x)}{f(y)} ,$$

where f and ϕ are reciprocal functions.

Solving separable DEs:

The algorithm is very simple, but magic: treat dy/dx as a quotient of differentials, and multiply through to rewrite the DE as

$$f(y) dy = g(x) dx .$$

Then antidifferentiate the left side with respect to y and the right side with respect to x :

$$\int f(y) dy = \int g(x) dx .$$

If $F(y)$ and $G(x)$ are antiderivatives of $f(y)$ and $g(x)$, respectively, and if you collect the constants of integration on one side of the equation, then this yields an equation involving the solution function y , and the variable x :

$$F(y) = G(x) + C .$$

This equation defines y implicitly as a function of x . Sometimes you can use algebra to explicitly solve for y as a function of x . Whether or not you can find the explicit formula, the constant C can be adjusted to solve initial value problems.

Mathematical justification for the method of separating differentials:

(We went through this on Wednesday) The use of differentials is disguising an application of the chain rule. Here is the explanation for the magic method: The differential equation

$$\frac{dy}{dx} = \frac{g(x)}{f(y)}$$

can be rewritten without differential magic, as

$$f(y) \left[\frac{dy}{dx} \right] = g(x).$$

If $y(x)$ is any solution to this rewritten equation, then the left side, namely

$$f(y(x)) \left[\frac{dy}{dx} \right]$$

is the derivative with respect to x of

$$F(y(x)) ,$$

where $F(y)$ is any antiderivative of $f(y)$ (with respect to y). This is just the chain rule! Thus if $G(x)$ is any antiderivative of $g(x)$ (w.r.t.x), we can legally antidifferentiate the rewritten DE with respect to x (on both sides) to get

$$F(y(x)) = G(x) + C$$

which is what we got by differential magic before! (This is actually the same magic you have been using whenever you used the method of "u substitution" to compute complicated antiderivatives...you've probably stopped thinking a long time ago about why this works.)

Now dig out your Wednesday notes.

Exercise 3, page 3 Wednesday Jan 13 notes: Do parts 3a), 3b), 3c) with all that open space on Wednesday's notes. Since I'm in Maple right now, I've put in the commands to check some of your hand work:

3a) "dsolve", a command in the "DEtools" library, will try to solve differential equations or initial value problems symbolically.

3b) "DEplot" will draw a slope field and specified solution trajectories; it not as easy or pretty as the non-Maple applet "dfield", but since we're in Maple now, I'll use DEplot.

3c) ?

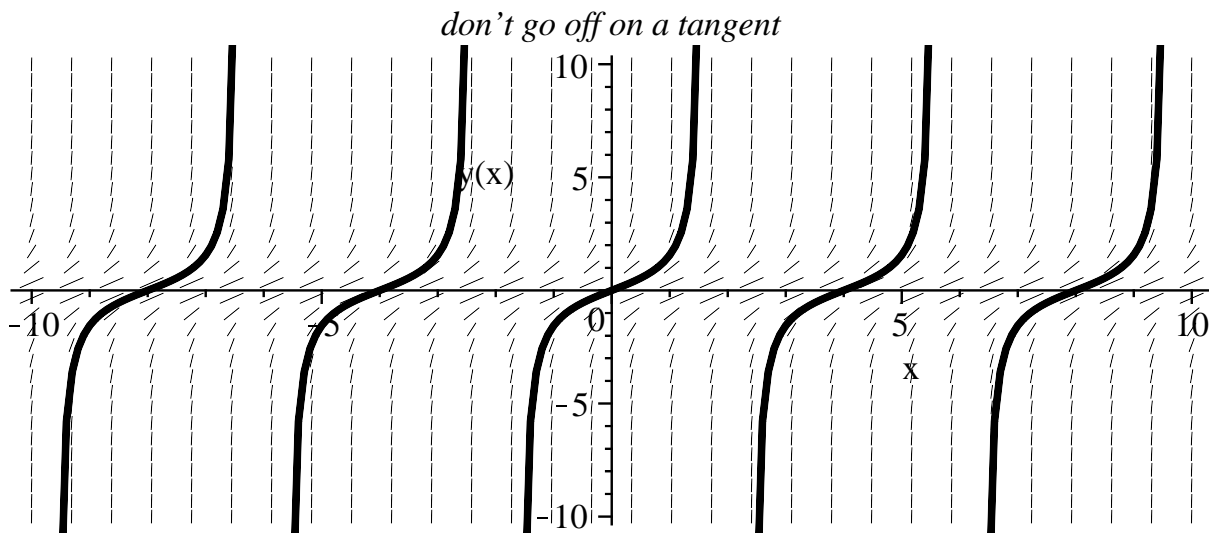
Here's how you can use Maple to check your answer (once you find the right commands):

```
> with(DEtools): #a package of differential equation commands
> dsolve([diff(y(x),x)=1+y(x)^2,y(0)=y[0]]);
          y(x) = tan(x + arctan(y_0))
```

(1)

Maple slope fields and solution graphs:

```
> deqtn:=diff(y(x),x)=1+y(x)^2: #this is the DE
DEplot(deqtn,y(x),-10..10,[y(-8)=0],[y(-4)=0],
[y(0)=0],[y(4)=0],[y(8)=0]},
y=-10..10,arrows=line, color=black,linecolor=black,
dirgrid=[30,30],stepsize=.1, title='don't go off on a tangent');
```



Exercise 4, page 4 Wednesday notes. On Wednesday we used separation of variables to find solutions to

$$\frac{dy}{dx} = y^{2/3}.$$

When we solved the resulting implicit equation for $y(x)$ we got an explicit formula equivalent to

$$y(x) = \frac{1}{27} (x - k)^3.$$

The graphs $y = y(x)$ of these solution functions are all horizontal translations (k units in the positive x -direction), of $y = \frac{1}{27} x^3$. By adjusting the translation parameter k , we can solve any initial value problem we want.

At the very end of class I pointed out that there was an extra solution we did not get from separation of variables, namely the zero function $y(x) = 0$, which certainly solves the DE since for the zero function both sides of the DE yield 0. How could we have missed this solution when we did separation of variables? Well, we divided both sides of the equation by $y^{2/3}$, which is not mathematically valid when $y = 0$! Thus, it's exactly along the line $y=0$ (the x -axis), that strange things can happen.

Solutions to separable differential equations which exist but which are not accounted for when you apply the separation of variables algorithm are called **singular solutions**. So in this example, the zero function is a singular solution. By defining piecewise functions we can get infinitely many singular solutions to this DE.

4b) Discuss how you could use the bracket notation to define a singular solution to this DE which followed a cubic for $x \leq -1$, was identically zero for $-1 \leq x \leq 2$, and followed another cubic for $2 \leq x$...the graph of this function is on the next page.

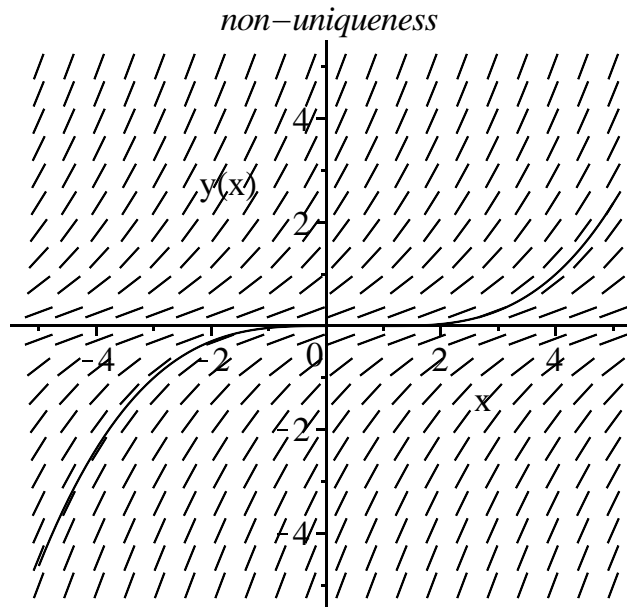
Maple check:

```
> diffeqtn4:=diff(y(x),x)=y(x)^(2/3):  
  dsolve(diffeqtn4,y(x));
```

$$y(x)^{1/3} - \frac{1}{3}x - _C1 = 0 \quad (2)$$

```
>
```

Silly Maple didn't find the singular solutions! Here's a graph of the singular solution you're writing the formula for in 4b: It's one of infinitely many solutions to the IVP for $y(0) = 0$.



Exercise 5: Discuss the existence–uniqueness theorem on page 5 of Wednesday notes, and discuss how many of its facets are illustrated in Exercises 3 and 4. There are a lot of interesting questions to discuss.