Name: Solutions

I.D. number:

Math 2280–1
FINAL EXAM
May 6, 2010

This exam is closed–book and closed–note. You may use a scientific calculator, but not one which is capable of graphing or of solving differential or linear algebra equations. Laplace Transform and integral tables are included with this exam. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** This exam counts for 30% of your course grade. It has been written so that there are 150 points possible, and the point values for each problem are indicated in the right–hand margin. **Good Luck!**

problem	score	possible
1	_______	20
2	_______	35
3	_______	20
4	_______	20
5	_______	35
6	_______	10
7	_______	10
total	_______	150

1a) What is the general solution to the unforced, undamped oscillator differential equation

$$\frac{d^2}{dt^2}x(t)+\omega_0^2 x(t)=0\ ?$$

You may just exhibit the solution if you recall it with no work. (6 points)

basis $\{\cos\omega_0 t, \sin\omega_0 t\}$

$$x(t)=A\cos\omega_0 t+B\sin\omega_0 t$$

1b) Use Laplace transform techniques, and possibly your work from (1a), to find the general solution to the undamped forced oscillator equation with resonance:

$$\frac{d^2}{dt^2}x(t)+\omega_0^2 x(t)=F_0\cos(\omega_0 t).$$

(7 points)

$\mathcal{L}$: let $x_0, v_0=0$.

then

$$s^2X(s)+\omega_0^2X(s)=F_0\frac{s}{s^2+\omega_0^2}$$

$$X(s)\cancel{(s^2+\omega_0^2)}=F_0\frac{s}{(s^2+\omega_0^2)^2}$$

from table, $k=\omega_0$

$$\boxed{x(t)=\underbrace{\frac{F_0}{2\omega_0}t\sin\omega_0 t}_{x_p}+\underbrace{A\cos\omega_0 t+B\sin\omega_0 t}_{x_H}}$$

1c) Use the method of undetermined coefficients and part (1a) to exhibit the general solution to the non–resonant undamped forced oscillator equation

$$\frac{d^2}{dt^2}x(t)+\omega_0^2 x(t)=F_0\cos(\omega t)$$

$$\omega\neq\omega_0$$

(7 points)

$\omega_0^2($ try $x_p=A\cos\omega t$ since $\omega\neq\omega_0$

$x_p'=-A\omega\sin\omega t$

$+\ 1($ $x_p''=-A\omega^2\cos\omega t$

$$x_p''+\omega_0^2x_p=\cos\omega t\,(A\omega_0^2-A\omega^2)\overset{?}{=}F_0\cos\omega t$$

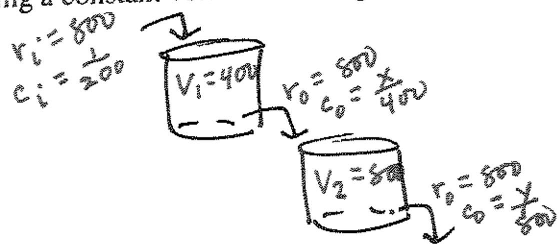
$$A(\omega_0^2-\omega^2)=F_0$$

$$A=\frac{F_0}{\omega_0^2-\omega^2}$$

so

$$\boxed{x(t)=\frac{F_0}{\omega_0^2-\omega^2}\cos\omega t+A\cos\omega_0 t+B\sin\omega_0 t}$$

2) We first encountered a tank cascade in Chapter 1, and this should be the last time we encounter one for at least several months. Consider two tanks. Salt water flows into the first tank at a constant rate of 800 gallons an hour, with a concentration of 1 pound of salt per 200 gallons of water. This first tank maintains a constant volume of 400 gallons by continuously pumping well–mixed water into the second tank, at the same constant rate of 800 gallons/hour. The second tank pumps well–mixed water out at this same rate, maintaining a constant volume of 800 gallons.



2a) Let $x(t), y(t)$ denote the amount of salt in tanks 1 and 2, respectively. Show that this system is modeled by the differential equations

$$\frac{dx}{dt} = -2x + 4$$
$$\frac{dy}{dt} = 2x - y$$

(5 points)

$$\frac{dx}{dt} = r_i c_i - r_o c_o = \frac{800}{200} - \frac{800\,x}{400} = 4 - 2x$$

$$\frac{dy}{dt} = r_i c_i - r_o c_o = 800\frac{x}{400} - 800\frac{y}{800} = 2x - y$$

2b) Suppose that the water in both tanks is initially pure, $x(0) = y(0) = 0$. Use the integrating factor methods of Chapter 1 for linear differential equations to solve the first differential equation above for $x(t)$, then plug that solution into the second differential equation and find $y(t)$.

(10 points)

$$\begin{cases} \frac{dx}{dt} = -2x + 4 \\ x(0) = 0 \end{cases}$$

$$\frac{dx}{dt} + 2x = 4$$
$$e^{2t}(x' + 2x) = 4e^{2t}$$
$$(e^{2t}x)' = 4e^{2t}$$
$$e^{2t}x = \int 4e^{2t}\,dt = 2e^{2t} + C$$
$$x = 2 + Ce^{-2t}$$

$x(0) = 0$ so $C = -2$

$$\boxed{x = 2 - 2e^{-2t}}$$

$$\begin{cases} \frac{dy}{dt} = 2(2 - 2e^{-2t}) - y \\ y(0) = 0 \end{cases}$$

$$y' + y = 4 - 4e^{-2t}$$
$$(e^t y)' = e^t(4 - 4e^{-2t}) = 4e^t - 4e^{-t}$$
$$e^t y = \int \text{RHS}\,dt = 4e^t + 4e^{-t} + C$$

$C = -8$ since $y_0 = 0$

$$\boxed{y = 4 + 4e^{-2t} - 8e^{-t}}$$

2c) Resolve the same initial value problem

$$\frac{dx}{dt} = -2x + 4$$
$$\frac{dy}{dt} = 2x - y$$
$$x(0) = y(0) = 0$$

using Laplace transform. (10 points)

$$s X(s) = -2X(s) + 4/s$$
$$s Y(s) = 2X(s) - Y(s)$$

$$X(s)(s+2) = 4/s \quad \text{so} \quad X(s) = \frac{4}{(s+2)(s)}$$
$$Y(s)(s+1) = 2X(s)$$
$$\text{so} \quad Y(s) = \frac{2X(s)}{s+1} = \frac{8}{(s+2)s(s+1)}$$

$$X(s) = \frac{4}{(s+2)(s)} = 2\left(\frac{1}{s} - \frac{1}{s+2}\right) \qquad \text{so} \quad \boxed{x(t) = 2 - 2e^{-2t}}$$

$$Y(s) = \frac{8}{(s+2)s(s+1)} = \frac{A}{s+2} + \frac{B}{(s+1)} + \frac{C}{s}$$

$$\cancel{8 = A(s+2)s + Bs + C(s+2)^2}$$
$$\cancel{@\, s=0: \quad 8 = 4C \Rightarrow C = 2}$$
$$\cancel{@\, s=-2: \quad 8 = -2B \Rightarrow B = -4}$$
$$\cancel{\text{coeff of } s^2: \quad 8 = A + C = A + 2 \Rightarrow A = 6}$$
$$\cancel{Y(s) = \frac{2}{s} + \frac{6}{s+2} - \frac{4}{(s+2)^2}}$$
$$\cancel{\text{so} \quad y(t) = 2 + 6e^{-2t}}$$

oops!

$$8 = As(s+1) + Bs(s+2) + C(s+1)(s+2)$$

$s=0$: $8 = 2C \Rightarrow C = 4$

$s=-1$: $8 = B(-1) \Rightarrow B = -8$

$s=-2$: $8 = A(-2)(-1) \Rightarrow A = 4$

$$\text{so} \quad Y(s) = \frac{4}{s+2} - \frac{8}{s+1} + \frac{4}{s}$$

$$\boxed{y(t) = 4e^{-2t} - 8e^{-t} + 4}$$

2d) Resolve this initial value problem

$$\frac{dx}{dt} = -2x + 4$$
$$\frac{dy}{dt} = 2x - y$$
$$x(0) = y(0) = 0$$

one more time by rewriting it as a nonhomogeneous system in matrix–vector form, and using the matrix theory for first order systems of differential equations (i.e. find general homogeneous solutions via eigenvector analysis, then find a particular solution, then solve IVP). Notice that you can find a particular solution by considering what happens to the salt amounts as time approaches infinity.

(10 points)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 0 \\ 2 & -1-\lambda \end{vmatrix} = (\lambda+2)(\lambda+1) \Rightarrow \lambda = -1, -2$$ (diag'l entries for Δular matrix)

$\lambda = -1$

$$\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\lambda = -2$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 2 & 1 & 0 \end{array}\right]$$

$$\vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\vec{x}_H = c_1 e^{-t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

$\vec{x}_p = \vec{c}$ yields

$$\vec{x}_p' = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$0 = -2c_1 + 4 \Rightarrow c_1 = 2$$
$$0 = 2(2) - c_2 \Rightarrow c_2 = 4.$$

$$\vec{x}_p = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

(as $t \to \infty$, $\vec{x}(t) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ since $c_i = \frac{1}{200}$ and $V_1 = 400$, $V_2 = 800$)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + c_1 e^{-t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

so
$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} -8 \\ 2 \end{bmatrix}.$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} - 8e^{-t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2e^{-2t}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 - 2e^{-2t} \\ 4 - 8e^{-t} + 4e^{-2t} \end{bmatrix}$$

3) Find the matrix exponentials for the following matrices.

3a)

$$A = \begin{bmatrix} -2 & 0 \\ 2 & -1 \end{bmatrix}$$

Hint: You may use your work from (2d). (10 points)

from (2d) $\quad \Phi(t) = \begin{bmatrix} 0 & -e^{-2t} \\ e^{-t} & 2e^{-2t} \end{bmatrix}$

$$e^{At} = \Phi(t)\Phi(0)^{-1} = \begin{bmatrix} 0 & -e^{-2t} \\ e^{-t} & 2e^{-2t} \end{bmatrix} \frac{1}{1} \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2t} & 0 \\ 2e^{-t}-2e^{-2t} & e^{-t} \end{bmatrix}$$

3b)

$$B = \begin{bmatrix} 0 & -8 \\ 2 & 0 \end{bmatrix}$$

(10 points)

$|B-\lambda I| = \lambda^2+16$

$\lambda = \pm 4i$

$\lambda = 4i$:

$$\begin{array}{l} \\ \\ R_2/2 \\ -R_1/4 \\ \\ -iR_1+R_2 \end{array} \left[\begin{array}{cc|c} -4i & -8 & 0 \\ 2 & -4i & 0 \\ \hline 1 & -2i & 0 \\ i & 2 & 0 \\ \hline 1 & -2i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{v} = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$

$$e^{4it}\begin{bmatrix} 2i \\ 1 \end{bmatrix} = (\cos 4t + i\sin 4t)\begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

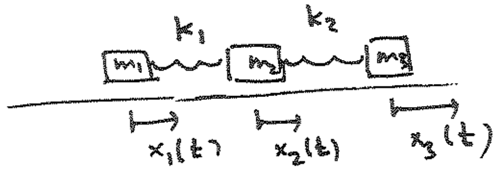
$$= \begin{bmatrix} -2\sin 4t & +2i\cos 4t \\ \cos 4t & +i\sin 4t \end{bmatrix}.$$

from lower left,

$$\Phi(t) = \begin{bmatrix} 2\cos 4t & -2\sin 4t \\ \sin 4t & \cos 4t \end{bmatrix}$$

so $\quad e^{Bt} = \begin{bmatrix} \cos 4t & -2\sin 4t \\ \frac{1}{2}\sin 4t & \cos 4t \end{bmatrix}.$

4a) Consider the 3 mass and two spring "train" indicated below, with masses, spring Hooke's constants, and displacements from equilibrium as labeled. Derive the second order system of differential equations for this no–drag train. (6 points)



$$m_1 x_1'' = k_1(x_2 - x_1)$$
$$m_2 x_2'' = -k_1(x_2 - x_1) + k_2(x_3 - x_2)$$
$$m_3 x_3'' = -k_2(x_3 - x_2)$$

4b) Assume all spring constants are equal to 10 Newtons/meter. What should the three masses be (in kg) so that the system above reduces to

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \\ x_3''(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 2 & -4 & 2 \\ 0 & 4 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} ?$$

(4 points)

$$x_1'' = \frac{10}{m_1}(x_2 - x_1)$$
$$x_2'' = \frac{10}{m_2} x_1 - \frac{20}{m_2} x_2 + \frac{10}{m_2} x_3$$
$$x_3'' = \frac{10}{m_3} x_2 - \frac{10}{m_3} x_3$$

so $m_1 = 10$ kg

$m_2 = 5$ kg

m_3 : $\frac{10}{m_3} = 4$; $m_3 = 5/2$ kg.

4c) Maple to the rescue! Use the output below to write down the general solution to this system of three second order differential equations. Recall that the first column vector lists the eigenvalues, and the matrix contains the corresponding eigenbases in its columns. Also describe the three fundamental "modes" for this train.

> B := Matrix(3, 3, [-1, 1, 0, 2,-4, 2, 0, 4,-4]);

$$B := \begin{bmatrix} -1 & 1 & 0 \\ 2 & -4 & 2 \\ 0 & 4 & -4 \end{bmatrix} \tag{1}$$

> Eigenvectors(B);

$$\begin{bmatrix} -7 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{8} & -\frac{1}{2} & 1 \\ -\frac{3}{4} & \frac{1}{2} & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{2}$$

(10 points)

$\lambda = -7 \quad \omega = \sqrt{7}, \quad \vec{v} = \begin{bmatrix} 1 \\ -6 \\ 8 \end{bmatrix}$

$\lambda = -2 \quad \omega = \sqrt{2}, \quad \vec{v} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

$\lambda = 0 \quad (\omega = 0) \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$

$$\vec{x}(t) = (c_1 + c_2 t)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (c_3 \cos\sqrt{2}\,t + c_4 \sin\sqrt{2}\,t)\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + (c_5 \cos\sqrt{7}t + c_6 \sin\sqrt{7}t)\begin{bmatrix} 1 \\ -6 \\ 8 \end{bmatrix}$$

equal constant speed velocity, or fixed translation, for each car.

$\omega = \sqrt{2}$, first car is out of phase with second two

$\omega = \sqrt{7}$, fastest vibration, middle car is out of phase with the outer two

5) Consider the system of differential equations below which models two populations x(t) and y(t):

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 4x - 2xy \\ -4y + xy \end{bmatrix}$$

5a) If this was a model of two interacting populations, what kind would it be? Explain. (4 points)

predator prey. x = prey population, grows exponentially in absense of predator, which decreases growth rate

y = predator population, decays exponentially w/o prey, but can thrive with many prey present

5b) Find both equilbrium solutions to this system of differential equations. (There are only two!) (4 points)

$x(4-2y) = 0$

$y(-4+x) = 0$

$x = 0 \Downarrow y = 0$

$4-2y=0 \Downarrow y=2 \Downarrow x=4.$

$(0,0)$ $(4,2).$

5c) Find the linearization of the population model, near the equilibrium solution for which both populations are positive. The system if repeated below for your convenience. (6 points)

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 4x - 2xy \\ -4y + xy \end{bmatrix}$$

$\begin{bmatrix} x_e \\ y_e \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ $\quad J = \begin{bmatrix} 4-2y & -2x \\ y & -4+x \end{bmatrix}$ $\quad J\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{bmatrix} 0 & -8 \\ 2 & 0 \end{bmatrix}.$

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & -8 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

5d) Find the general solution $\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ to the linearized problem in 5c. What kind of equilibrium is $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, for this linear system? Explain why you are not able to deduce from this information whether or not the corresponding equilibrium solution for the non–linear problem is stable. (10 points)

found e^{Bt} for this matrix in 3b, so

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \cos 4t & -2\sin 4t \\ \frac{1}{2}\sin 4t & \cos 4t \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}.$$

sol'ns lie on ellipses $\frac{u^2}{4} + v^2 = \left(\frac{u_0^2}{4} + v_0^2\right)$

so $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a stable center.

since $Re(\lambda) = 0$ this is borderline case for non-linear stability, and non-linear problem could be any of stable or unstable spiral point, or stable center

5e) We have a method of analyzing the non–linear stability in borderline cases like this, which depends on separable differential equations. Use this method to determine whether or not our equilibrium solution for the non–linear problem is stable. (6 points)

$$\left.\begin{aligned} \frac{dx}{dt} &= x(4-2y) \\ \frac{dy}{dt} &= y(x-4) \end{aligned}\right\} \nearrow \quad \frac{dy}{dx} = \frac{y(x-4)}{x(4-2y)}$$

$$\frac{4-2y}{y}\,dy = \frac{x-4}{x}\,dx$$

$$\frac{4}{y} - 2\,dy = 1 - \frac{4}{x}\,dx$$

$x, y > 0$: $\quad 4\ln y - 2y = x - 4\ln x + C$

so solution trajectories lie on level curves of

$f(x,y) = 4\ln y + 4\ln x - 2y - x$

which has a global max at $(x,y) = (4,2)$

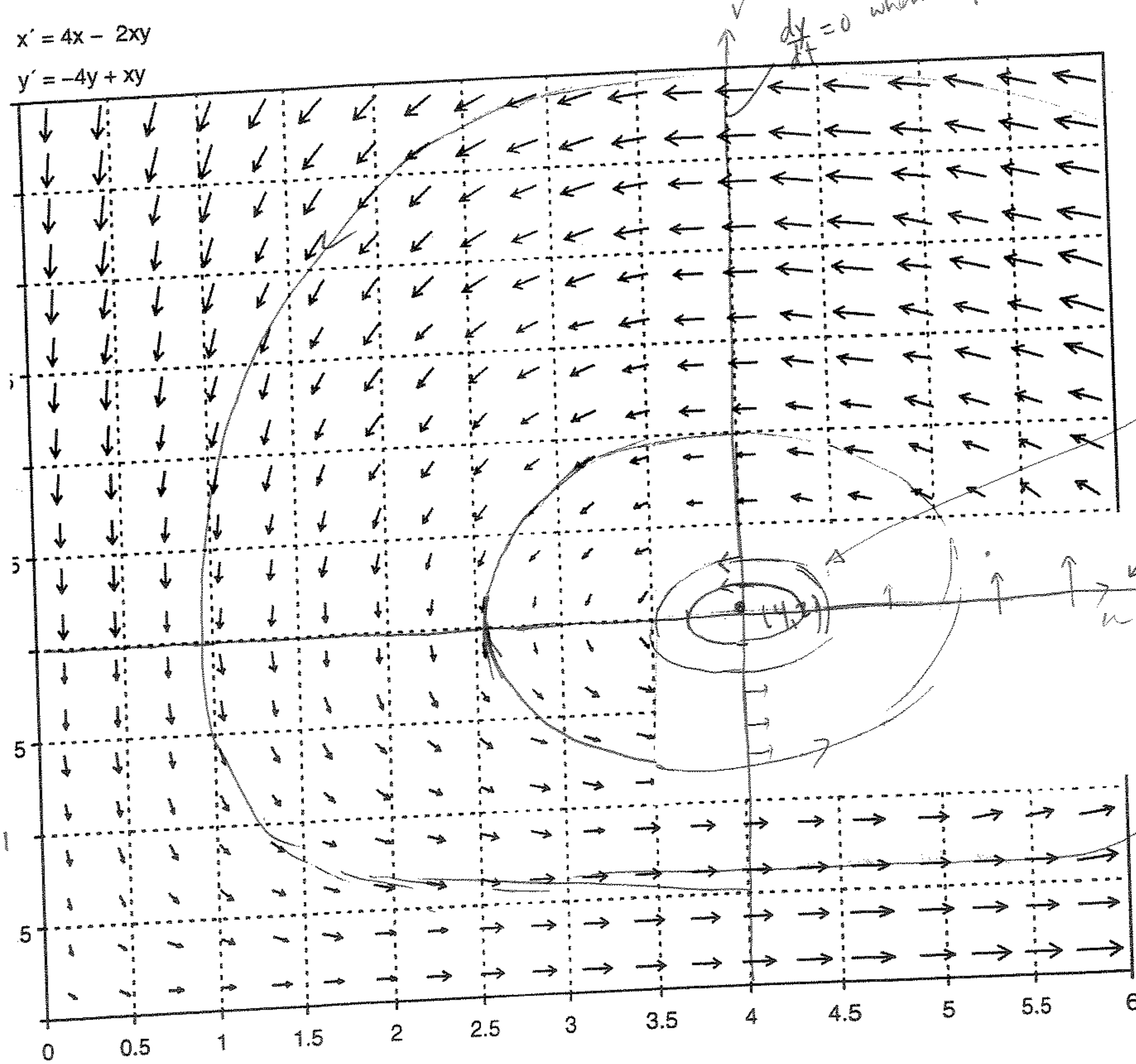
$f_x = \frac{4}{x} - 1 \qquad f_y = \frac{4}{y} - 2$

$f_{xx} = -4/x^2 \qquad f_{yy} = \frac{-4}{y^2}$

thus these level curves are all closed loops and nonlinear problem has stable center @ $(4,2)$

5f) Use your work from (5de) to fill the portion of the phase portrait below which has been excised. Then describe the behavior of solutions to this system of differential equations, in all cases when both initial populations are positive. Sketch typical sol'n trajectories (5 points)

$\frac{dy}{dt} = 0$ when $x = 4$



$\frac{dx}{dt} = 0$ when $y = 2$

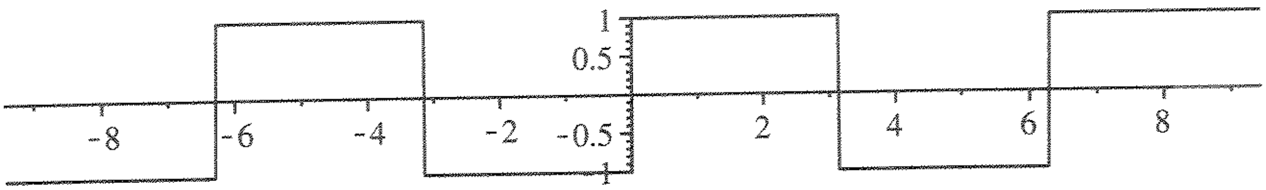
linearized ellipses $\frac{u^2}{4} + \frac{v^2}{1} = C$

close up!

6) We consider a 2 π–periodic square–wave function, given on the interval $[-\pi, \pi]$ by

$$f(t) = -1, \quad \text{for } t < 0$$
$$f(t) = 1, \quad \text{for } t > 0.$$

Here's a graph of a piece of this function:



Show the Fourier series for $f(t)$ is given by

$$f(t) \sim \frac{4\left(\sum_{n=odd} \frac{\sin(n\,t)}{n}\right)}{\pi}$$

(10 points)

f is odd so its Fourier series is a sine series

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$$

$$= \frac{2}{\pi}\underbrace{\int_0^{\pi} 1 \sin nt\,dt}_{-\frac{\cos nt}{n}} \qquad \text{since } f(t)\sin nt \text{ is even}$$

$$= \frac{2}{\pi n}\left[-\cos n\pi + 1\right] = \begin{cases} \frac{2}{\pi n}(-1+1) = 0 & \text{if } n \text{ even} \\ \frac{2}{\pi n}(-(-1)+1) = \frac{4}{\pi n} & n \text{ odd.} \end{cases}$$

$$\Rightarrow f \sim \sum_{n \text{ odd}} \frac{4}{\pi n}\sin nt$$

7) Consider the square wave function $f(t)$ from problem 6. Find a Fourier series particular solution to the forced oscillation problem

$$x''(t) + 4x(t) = f(t).$$

Discuss whether or not resonance occurs. (10 points)

$$f(t) \sim \sum_{n \text{ odd}} \frac{4}{\pi n} \sin nt$$

since $\omega_0 = 2$ is not an odd integer there will be no resonance.

$$x_p = \sum_{n \text{ odd}} a_n \sin nt$$

$$x_p'' + 4x_p = \sum_{n \text{ odd}} (-n^2 a_n + 4a_n) \sin nt = \sum_{n \text{ odd}} \frac{4}{\pi n} \sin nt$$

$$a_n(4-n^2) = \frac{4}{\pi n}$$

$$a_n = \frac{4}{\pi n(4-n^2)}$$

$$x_p(t) = \sum_{n \text{ odd}} \frac{4}{\pi n(4-n^2)} \sin nt$$

(full soln is $x = x_p(t) + A\cos 2t + B\sin 2t$)