Math 2280-1
Friday Feb. 5

HW for Friday Feb. 12

3.1 1, 2, 4 5, 11, 13, 17 27, 29, 30, 31, 33, 34, 35

3.2 2 5 9 11 13 21 22 25 26

3.3 3, 10, 14 21 22, 29, 33, 37

3.4 4, 5, 6 13, 15, 19, 23

- Maple project questions?

- $A\cos\omega t + B\sin\omega t = C\cos(\omega t-\alpha)$ Jan 27 notes

  $= C\cos(\omega(t-\delta))$

  linear combo form   Amplitude-phase form

- harvesting logistic fish? Feb 1 notes

Then, back to Chapter 3!

Recall, on Wednesday we discussed Theorems 1, 2, 3.

(We proved 1 & 3, and will come to understand why 2 is true, later)

Theorem 1 $L(y) := a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y$ s.t. each a_j cont. on fixed interval I

then $L: C^n(I) \to C(I)$ is linear

$\{y(x) \mid y, y', \dots y^{n} \text{ are cont. on } I\}$

$\{z(x) \text{ s.t. } z \text{ cont on } I\}$.

Theorem 2 $L(y) := y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y$, a_j cont on I $j = 0 \cdots n-1$

$x_0 \in I$, $f \in C(I)$, $\vec{b} \in \mathbb{R}^n$

then, $\exists!$ sol'n to IVP, defined on all of I, i.e. $y \in C^{(n)}(I)$.

$$\text{IVP} \begin{cases} L(y) = f \\ y(x_0) = b_0 \\ y'(x_0) = b_1 \\ \vdots \\ y^{(n-1)}(x_0) = b_{n-1} \end{cases}$$

– proof deferred! –

Theorem 3 $L: V \to W$ linear, $b \in W$.

then if $L(v_p) = b$, then the solution set to $L(v) = b$ is

(page 4 Wed)

$$\{v = v_p + v_H \text{ s.t. } L(v_H) = 0\}.$$
$$= \{v = v_p + v_H \text{ s.t. } v_H \in \ker(L)\}.$$

the "H" stands for "homogeneous sol'n", since $L(v) = 0$ is called the homogeneous equation

Exercise 1: Use these theorems to show why every sol'n to

$$y'' - 4y = 1 \qquad (\text{for } x \in \mathbb{R})$$

is of the form $y = -\frac{1}{4} + c_1 e^{2x} + c_2 e^{-2x}$ (this was example 3 on Wed.)

<u>Theorem</u> 4: Let $L(y)$ be the n^{th} order linear operator defined on page 1.
Then the dimension of $\ker(L) = n$.

(so if we can find n linearly ind. sol'ns to $L(y)=0$, they will be a basis for $\ker(L)$, i.e. every sol'n will be a linear combo of the basis sol'ns,

$$y_H = c_1 y_1 + c_2 y_2 + \dots + c_n y_n \quad)$$

<u>proof</u>:

If $y(x) \in V$

write $\vec{IV}(y) := \begin{bmatrix} y(x_0) \\ y'(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix}$

By the $\exists!$ theorem for IVP, $\exists$ sol'ns $y_1, y_2, \dots y_n$ to $L(y)=0$

such that

$$\vec{IV}(y_j) = \vec{e}_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{entry}_j$$

$\boxed{\text{Show } \{y_1, \dots y_n\} \text{ are a basis:}}$

- These sol'ns are <u>linearly independent</u>, because if

$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n \equiv 0 \qquad (\forall x \in I)$$

$$\frac{d}{dx} \Rightarrow \quad c_1 y_1' + c_2 y_2' + \dots + c_n y_n' \equiv 0$$

$$\frac{d^2}{dx^2} \Rightarrow \quad c_1 y_1'' + c_2 y_2'' + \dots + c_n y_n'' \equiv 0$$

$$\vdots$$

$$c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)} + \dots + c_n y_n^{(n-1)} \equiv 0$$

called the Wronskian matrix of $y_1, \dots y_n$; its det is called Wronskian $\longrightarrow$

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0} \qquad \forall x \in I$$

In particular, at $x = x_0$, the Wronskian matrix (above) is the identity, so $\vec{c} = \vec{0}$ ■ (shows linear independence)

- These solutions <u>span</u> $\ker(L)$ because if $z(x)$ solves $L(z)=0$, then

for $\begin{bmatrix} z(x_0) \\ z'(x_0) \\ \vdots \\ z^{(n-1)}(x_0) \end{bmatrix} := \vec{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}$, the (other?) sol'n $y(x) = b_0 y_1(x) + b_1 y_2(x) + \dots + b_{n-1} y_n(x)$

has the same initial value vector!, i.e. $\vec{IV}(y) = \vec{b}$.

So, by uniqueness (in $\exists!$ theorem), $z \equiv y$! ■

Exercise 2 Find all solutions to

$$y''' + y'' - 30y' = 0$$

Hint: try for a basis of this 3-dim'l vector space made out of exponential functions, $y = e^{rx}$.

Exercise 3 Find all soltns to

$$y''' + y'' - 30y' = 56e^x$$

Hint: try for a y_P of the form $y_P = Ce^x$.