

Math 2280-1

Wed Feb. 3

(1)

- Let's do "harvesting fish" example from Monday.
- Then begin chapter 3 below! (3.1-3.2)

Linear differential equations of n^{th} order

Def: $L(y) := a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y$

is called a linear differential operator of order n

We assume the functions $a_j(x)$, $0 \leq j \leq n$ are defined and continuous for $x \in I$, some interval in $\mathbb{R}$.

Theorem 1 L is a linear transformation

from the vector space of n -times continuously differentiable functions $y(x)$ defined for $x \in I$ ("V")
with codomain equal to the continuous functions on I ("W")

proof: This amounts to showing

$$L(y_1 + y_2) = L(y_1) + L(y_2)$$

$$L(cy_1) = cL(y_1)$$

$$\forall y_1, y_2 \in V \\ c \in \mathbb{R}$$

or more compactly,

$$L(c_1 y_1 + c_2 y_2) = c_1 L(y_1) + c_2 L(y_2)$$

proof:

examples: $L(y) = y' + p(x)y$ 1st order

$L(y) = y'' + p(x)y' + q(x)y$ 2nd order

Def Let the $a_j(x)$, I be as on page 1.

Then an n^{th} order linear differential eqn is an DE of the form

$$L(y) = f(x)$$

where f is continuous on I .

e.g. 1st order: $y' + p(x)y = q(x)$

2nd order: $y'' + p(x)y' + q(x)y = f(x)$

etc.

Theorem 2: Let $I \subset \mathbb{R}$ an interval

Let $x_0 \in I$

Let $a_0(x), a_1(x), \dots, a_{n-1}(x), f(x)$ continuous on I

Then for any $b_0, b_1, \dots, b_{n-1} \in \mathbb{R}$

$\exists!$ solution to the initial value problem

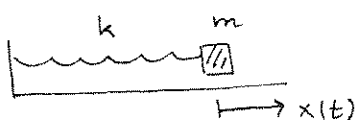
$$\begin{cases} L(y) := y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = f(x) & \forall x \in I \\ y(x_0) = b_0 \\ y'(x_0) = b_1 \\ y''(x_0) = b_2 \\ \vdots \\ y^{(n-1)}(x_0) = b_{n-1} \end{cases}$$

(We'll discuss proof later in course)

Examples: $n=1$: $\begin{cases} y'(x) + p(x)y = q(x) \\ y(x_0) = y_0 \end{cases}$

We did this b1.5!!

$n=2$ example: oscillating spring IVP



$\begin{array}{l} \text{Hooke's const} \swarrow \text{coeff of} \searrow \text{external} \\ \downarrow \text{force} \quad \downarrow \text{force} \end{array}$

$$m x'' = \text{net forces} = -kx - cx' + f(t)$$

$$\begin{cases} x'' + \frac{c}{m}x' + \frac{k}{m}x = \frac{1}{m}f \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

if we specify initial displacement and velocity of spring configuration we expect unique sol'n $t \geq 0$ (repeatable experiment!)

plausibility argument for Theorem 2:

Assume all functions are ∞ 'ly diff'ble and that $y(x)$ has a convergent

$$\downarrow$$

$$a_j(x), f(x), y(x)$$

Taylor series at x_0 ,

$$y(x) = y(x_0) + y'(x_0)(x-x_0) + \frac{y''(x_0)}{2!}(x-x_0)^2 + \dots$$

$$= \sum_{m=0}^{\infty} \frac{y^{(m)}(x_0)}{m!} (x-x_0)^m$$

From IVP we know coeffs at start of series

$$y(x_0) = b_0$$

$$y'(x_0) = b_1$$

$$\vdots$$

$$y^{(n-1)}(x_0) = b_{n-1}$$

from DE, $y^{(n)}(x_0) = f(x_0) - a_0(x_0)b_0 - a_1(x_0)b_1 - \dots - a_{n-1}(x_0)b_{n-1}$

If we take $\frac{d}{dx}$ of DE, we can then solve for $y^{(n+1)}(x_0)$

then $\frac{d^2}{dx^2}$ of DE to get $y^{(n+2)}(x_0)$

recursively get all derivs of $y(x)$ at $x=x_0$, so the Taylor series for $y(x)$.!

Theorem 3 (Linear algebra)

Let $L: V \rightarrow W$ be a linear transformation.

Consider the problem of finding all solutions $\vec{v} \in V$ to the equation

$$L\vec{v} = \vec{b}$$

Let $\vec{v}_p$ be a (particular) solution to this equation.

Then every solution $\vec{v}$ can be written as

$$\vec{v} = \vec{v}_p + \vec{v}_H$$

where $\vec{v}_H$ is in the kernel of L , (i.e. solves the homogeneous equation $L\vec{v}_H = \vec{0}$)

(and each such $\vec{v} := \vec{v}_p + \vec{v}_H$ solves the original problem!)

proof: Let $\vec{v} = \vec{v}_p + \vec{v}_H$ where $L(\vec{v}_p) = \vec{b}$ and $L(\vec{v}_H) = \vec{0}$

$$\text{Then } L(\vec{v}) = L(\vec{v}_p + \vec{v}_H)$$

$$= L(\vec{v}_p) + L(\vec{v}_H) = \vec{b} + \vec{0} = \vec{b}, \text{ so } \vec{v} \text{ is a sol'n.}$$

Now let $\vec{w}$ be any sol'n, i.e. $L(\vec{w}) = \vec{b}$

Then

$$\vec{w} = \vec{v}_p + (\vec{w} - \vec{v}_p)$$

$$\vec{v}_H; \text{ note } L(\vec{w} - \vec{v}_p) = L(\vec{w}) - L(\vec{v}_p) = \vec{b} - \vec{b} = \vec{0}$$

so $\vec{w}$ is the sum of $\vec{v}_p$ with a vector in the kernel of L

example 1: $L: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ $L \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Solve $L \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 1 & -2 & 3 & 1 \\ \hline 1 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}$$

$$\begin{aligned} x_3 &= t \\ x_2 &= s \\ x_1 &= 2s - 3t + 1 \end{aligned}$$

$$\vec{x} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{x}_p} + s \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{x}_H} + t \underbrace{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}_{\vec{x}_H}$$

example 2:

$$L(y) := y' + 2y$$

solve $y' + 2y = 3$ for $y(x), x \in \mathbb{R}$

$$(e^{2x}y)' = e^{2x}3$$

$$e^{2x}y = \frac{3}{2}e^{2x} + C$$

$$y = \underbrace{\frac{3}{2}}_{\vec{y}_p} + \underbrace{Ce^{-2x}}_{\vec{y}_H}$$

For higher order DE's we will try to find the general solution by finding a y_p , and the general homogeneous sol'n

example 3 Explain why all sol'n's to

$$y'' - 4y = 1 \quad (\text{for } x \in \mathbb{R})$$

are of the form

$$y(x) = \frac{1}{4} + c_1 e^{2x} + c_2 e^{-2x}$$

hint: after finding y_p , and at least part of y_H ,

use $\exists!$ theorem to show every sol'n has claimed form.