Math 2280-1
Wednesday Feb. 24

postpone §3.6 # 13,16 until next week HW (because they're about damped forced oscillators, which we won't discuss until Friday). But do the other §3.6 exercises.

- Discuss variation of parameters method for finding y_p, Monday notes page 3
- do example of var pars, Tuesday notes page 3

That will complete §3.5.

§3.6: Forced oscillations and resonance.

On Tuesday we solved the undamped forced oscillator problem.

$$\begin{cases} mx'' + kx = F_0 \cos\omega t \quad \rightsquigarrow \quad x'' + \omega_0^2 = \frac{F_0}{m}\cos\omega t \qquad \omega_0 = \sqrt{\frac{k}{m}} \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

Case 1 $\omega \neq \omega_0$

$$x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}(\cos\omega t - \cos\omega_0 t) + x_0 \cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$$

Case 2 $\omega = \omega_0$

$$x(t) = \frac{F_0}{2m\omega_0}\, t \sin\omega t + x_0 \cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$$

the transition as $\omega \to \omega_0$ is interesting, and involves a phenomenon known as beating important e.g. to musicians.

Case 1a (boring) $|\omega-\omega_0|$ not small!

then from page 1, the sol'n to any IVP is of the form

$$x(t) = C_1 \cos\omega t + C_2 \cos(\omega_0 t - \alpha)$$

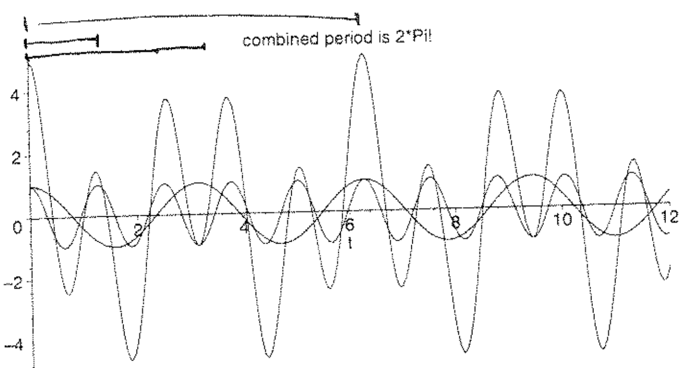
period $T = \frac{2\pi}{\omega}$ period $T_0 = \frac{2\pi}{\omega_0}$

this superposition may or may not be periodic, depending on whether T and T_0 have a common (integer) multiple (iff T/T_0 is rational!)

$\downarrow$ the least common such multiple would then be the (least) period of the superposition

period π period $2\pi/5$

```
> with(plots):
Warning, the name changecoords has been redefined
> plot({cos(2*t),cos(5*t),2*cos(2*t) +
  3*cos(5*t)},t=0..12,color=black,title=
  'combined period is 2*Pi!');
```



Case 1b (interesting) $|\omega-\omega_0|$ is small, but non-zero

write $\bar{\omega} = \frac{\omega+\omega_0}{2}$, $\delta = \frac{\omega-\omega_0}{2}$ $\Bigg\}$ so $\omega = \bar{\omega}+\delta$, $\omega_0 = \bar{\omega}-\delta$

From addition angle,

$$\cos((\bar{\omega}+\delta)t) - \cos((\bar{\omega}-\delta)t)$$

$$= \cancel{\cos\bar{\omega}t\cos\delta t} - \sin\bar{\omega}t\sin\delta t - \left(\cancel{\cos\bar{\omega}t\cos\delta t} + \sin\bar{\omega}t\sin\delta t\right)$$

$$= \underbrace{(-2\sin\delta t)}_{\text{slowing varying "amplitude" function with period } |2\pi/\delta|}\underbrace{(\sin\bar{\omega}t)}_{\text{oscillates with } \bar{\omega}\approx\omega\approx\omega_0}$$

Rewrite page 1 sol'n: (for case 1)

$$\boxed{x(t) = \frac{F_0}{m(\omega_0^2-\omega^2)}(-2)\sin\left(\frac{\omega-\omega_0}{2}t\right)\sin\left(\frac{\omega+\omega_0}{2}t\right) + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

This is called "beating", see example 2. p 215 (or Maple)

- Very important to musicians
- the transition to resonance!

as $\omega\to\omega_0$, and for $t\in[0,M]$,

$$\frac{\sin\left(\frac{\omega-\omega_0}{2}t\right)}{\left(\frac{\omega-\omega_0}{2}\right)t} \to 1$$

so $\frac{F_0}{m(\cancel{\omega_0-\omega})(\omega_0+\omega)}\cancel{2}\sin\left(\left(\frac{\cancel{\omega-\omega_0}}{\cancel{2}}\right)t\right)\sin\left(\frac{\omega+\omega_0}{2}t\right)$ Case 1

$$\longrightarrow \frac{F_0}{2m\omega_0}t\sin\omega_0 t$$ Case 2!

Example:

$F_0 = 5$ $\qquad$ $x_0 = 0$

$m = .1$ $\qquad$ $v_0 = 0$

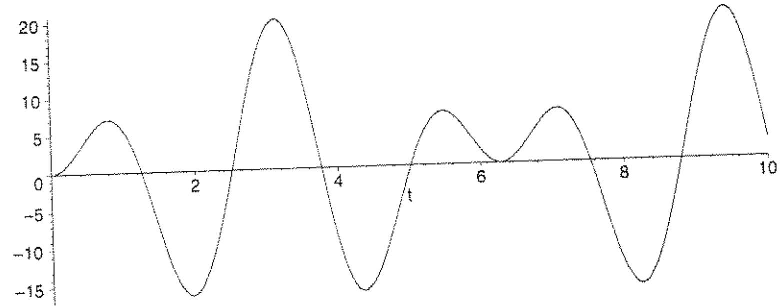
$w_0 = 3$ $\quad (k = .9)$

```
> plot(10*(cos(2*t)-cos(3*t)),t=0..10,color=black);
```

$w=2:$ $\quad \dfrac{F_0}{m}\dfrac{1}{w_0^2-w^2}\left[\cos wt - \cos w_0 t\right]$

$= \dfrac{50}{5}\left(\cos 2t - \cos 3t\right)$

(page 1 formula)



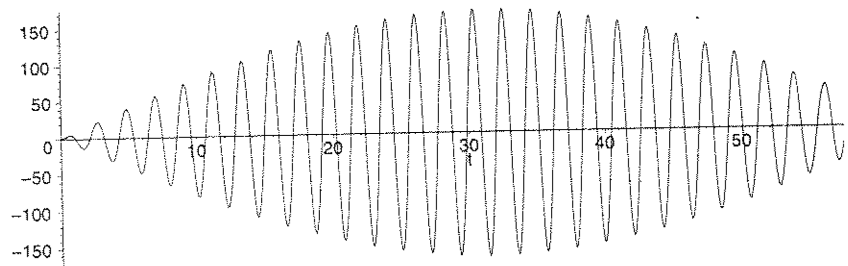

```
> plot(-100/(9-2.9^2)*sin(-.05*t)*sin(2.95*t),t=0..60,
  numpoints=200,color=black);
```

$w = 2.9$ $\qquad x(t) = \dfrac{50(-2)}{(9-(2.9)^2)}\sin(-.05t)\sin(2.95t)$

(page 2 formula)

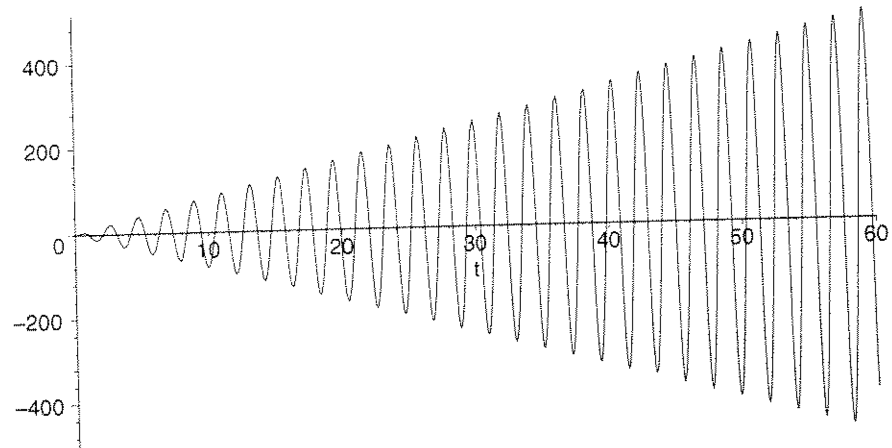
$T = \dfrac{2\pi}{.05} = 40\pi$

notice how large the time-varying "amplitude" is getting!




```
> plot(25/3*t*sin(3*t),t=0..60,numpoints=200,color=black);
```

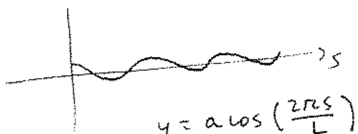
$w=3$ (page 1 formula) $\qquad x(t) = \dfrac{25}{3}t\sin 3t$



Wed. Case 3 $c \neq 0$!

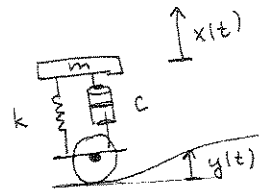
when c close to zero will also get interesting results.

unexpected
forced oscillation problems: p 218



$y = a\cos\left(\frac{2\pi s}{L}\right)$

road surface
period = L.



"car"

vertical acceleration:

$m x'' = -k(x-y) - c(x'-y')$

if $c = 0$

$m x'' + kx = ky$ if speed of car is v then $s = vt$

$= k a \cos\left(\frac{2\pi v}{L} t\right)$

is a forced oscillation problem.
trouble when

$\omega = \frac{2\pi v}{L} = \omega_0 = \sqrt{\frac{k}{m}}$

$v = \sqrt{\frac{k}{m}}\,\frac{L}{2\pi}$

(freeway)
"washboard"

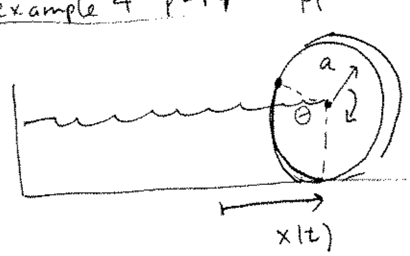
See also the front loading washing machine, #20 p. 222

Using total energy to deduce natural angular frequency:
(for conservative systems) KE + PE = const

1. $KE = \frac{1}{2} m v^2$ mass m, speed v (of center of mass)
2. $KE = \frac{1}{2} I \omega^2$ rotating object (ω = angular vel.) I = moment of intertia.
3. $PE = \frac{1}{2} k x^2$ stretched spring
4. $PE = mgh$ gravitational PE

total energy is additive

example 4 p 217



$E = \frac{1}{2} k x^2 + \frac{1}{2} I \omega^2 + \frac{1}{2} M x'(t)^2$

$x = a\theta$
$x' = a\theta' = a\omega$

I for solid wheel $= \frac{1}{2} M a^2$ (you can work this out!)

so

$E = \frac{1}{2} k x^2 + \frac{1}{4} M a^2 \left(\frac{x'}{a}\right)^2 + \frac{1}{2} M (x')^2$

$= \frac{1}{2} k x^2 + \frac{3}{4} M (x')^2$

$0 \equiv \frac{dE}{dt} = k x x' + \frac{3}{2} M x' x'' = x'\left[kx + \frac{3}{2} M x''\right]$

$\boxed{\frac{3}{2} M x'' + kx = 0}$ $\omega_0 = \sqrt{\frac{k}{\frac{3}{2}M}} = \sqrt{\frac{2k}{3M}}$