Math 2280-1
Tuesday Feb. 23 §3.5 & begin 3.6

- Finish Theorem 2 from yesterday. The statement (in my version) wasn't quite correct (power of x was one off).

corrected:

<u>Theorem 2</u> (Rule 2 page 205).

Let $L = D^n + a_{n-1}D^{n-1} + \dots + a_1 D + a_0 I$ be an n^{th} order, linear, constant coeff. operator.
Consider the problem of finding a particular sol'n y_p for

$L(y_p) = f$, i.e. $y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = f$.

~~Suppose $f \in V = \text{span}\{ e^{ax}\cos bx, e^{ax}\sin bx, xe^{ax}\cos bx, xe^{ax}\sin bx, \dots$~~

Suppose $f \in V$, where V is one of

$a=b=0$: $V = \text{span}\{1, x, \dots x^m\}$
$b=0$: $V = \text{span}\{e^{ax}, xe^{ax}, \dots x^m e^{ax}\}$
$a=0$: $V = \text{span}\{\cos bx, \sin bx, x\cos bx, x\sin bx, \dots x^m\cos bx, x^m \sin bx\}$
includes: $V = \text{span}\{e^{ax}\cos bx, e^{ax}\sin bx, xe^{ax}\cos bx, xe^{ax}\sin bx, \dots x^m e^{ax}\sin bx\}$

<u>Rule 1</u> was, if $r_1 = a+bi$ is <u>not</u> a root of charact poly $p(r)$ for L,
then $L: V \to V$ is isomorphism so $\exists!\, y_p \in V$ with $L(y_p) = f$

<u>Rule 2</u> is, if $r_1 = a+bi$ <u>is</u> a root of charact poly, with algebraic multiplicity $s > 1$
(so $p(r) = q(r)(r - r_1)^s$)

then let $W = \{ x^s g(x) \text{ s.t. } g \in V\}$
Then $\dim W = \dim V$ (multiply V basis fns by x^s to get W basis)
and $L: W \to V$ is isomorphism, so

$$\exists!\, y_p \in W \text{ s.t. } L(y_p) = f$$

<u>Proof</u> $L = q(D)(D - r_1 I)^s$

$$(D - r_1 I)^s (x^s e^{r_1 x}) = s!\, e^{r_1 x} \in V \quad \text{(with complex scalars)}$$
$$(D - r_1 I)^s (x^{s+1} e^{r_1 x}) = \frac{(s+1)!}{1!} x e^{r_1 x} \in V$$
$$\vdots$$
$$(D - r_1 I)^s (x^{s+m} e^{r_1 x}) = \frac{(s+m)!}{m!} x^m e^{r_1 x} \in V$$

deduce $L: W \to V$ is 1-1 & onto ☺

§3.6

Example 1 (Application), so $y(x) \rightsquigarrow x(t)$

Consider the forced undamped oscillator (§3.6), for displacement $x(t)$.

$$mx'' + kx = F_0 \cos\omega t \quad \rightsquigarrow \quad x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega t \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

↑ periodic (trig) forcing.

Case I $\omega \neq \omega_0$.

$V = \text{span}\{\cos\omega t, \sin\omega t\}$, $L: V \to V$ isomorphism, since $x_H = \text{span}\{\cos\omega_0 t, \sin\omega_0 t\}$
Rule 1

$x_p =$

ans $x = x_p + x_H$

$$= \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t + C\cos(\omega_0 t - \alpha)$$

Case II $\omega = \omega_0$

$x_p = A\,t\cos\omega t + B\,t\sin\omega t$, by Rule 2

ans: $x = x_p + x_H$

$$= \frac{F_0}{2m\omega_0}\,t\sin\omega_0 t + C\cos(\omega_0 t - \alpha)$$

- Now go back to Monday notes and discuss variation of parameters for y_P, which works for any f, once you know y_H — but it's harder than undetermined coef's

the var par example on page 4 is kind of messy.
It might be reasonable to find a y_P for

$$y'' + 2y' + y = e^{-x}$$

instead.
(We got a $y_P = \frac{1}{2}x^2 e^{-x}$ using Case 2 of undetermined coef's).