Hw for Fri Feb 26
3.5 3 4 17, 19, 36, 37, 43, 49
50, 51, 64
3.6 4, 5 7, 13, 16, 21, 22 20

Math 2280-1
Monday Feb. 22

§3.5 : Finding y_p for $Ly = f$, L const coeff linear operator.

$$L = D^{(n)} + a_{n-1}D^{(n-1)} + \dots + a_1 D + a_0 I. \qquad a_j \text{ const.}$$

Last Wednesday we worked examples consistent with:

Theorem 1 Suppose $f \in V$, a finite dimensional subspace of C^n(domain interval) of a
Suppose L (restricted to V) is an isomorphism.
Then $\exists!\ y_p \in V$ s.t. $L(y_p) = f$

proof: this is the definition of isomorphism. ∎

By the rank + nullity theorem we can deduce $L\big|_V$ is an isomorphism by checking its kernel = $\{\vec{0}\}$.
Then, if V has a basis

$$\{g_1, g_2, \dots g_k\}$$

finding $y_p = c_1 g_1 + \dots + c_k g_k$ is called the method of undetermined coeffs
because you need to find $c_1, c_2, \dots c_k$

Examples (p. 205)
↑
"guessing table"

- Finish working examples 3, 4 in Wednesday notes

typical V's closed with respect to linear differential operators

$V = \text{span}\{e^{rx}\}$

$V = \text{span}\{e^{ax}\cos bx, e^{ax}\sin bx\}$ $\qquad r = a \pm bi$

$V = \text{span}\{1, x, \dots x^k\}$ $\qquad r = 0$

$V = \text{span}\{e^{rx}, xe^{rx}, \dots x^k e^{rx}\}$.

What to do if $\ker L\big|_V \neq \{0\}$ for such a V? i.e when the r associated to V is a root of the characteristic polynomial $p(r)$ for L?

<u>Theorem 2</u> If V is a finite dimensional subspace as on page 1, and the exponential factor $r=r_1$ <u>is</u> a factor of $p(r)=(r-r_1)^{k_1}q(r)$ of algebraic multiplicity k_1

then let W be the subspace of C^n obtained by multiplying each element of V by x^{k_1+1}.

$$W=\{x^{k_1+1}g(x) \text{ s.t. } g\in V\}$$

Then $L:W\to V$ is an isomorphism.

(this explains the "fix it" recipe in page 205 table: multiply the guess you would've made by x^{k_1+1}).

<u>proof</u> $\dim W=\dim V$ (get basis of W by multiplying the V-basis functions by x^{k_1+1})

~~so~~ by construction

$$\ker(L|_W)=\{0\}.$$

& $\operatorname{image}(L|_W)\subset V$ because for $x^j e^{r_1 x}\in W$,

$$(D-r_1I)(x^j e^{r_1x}) = jx^{j-1}e^{r_1x} + \cancel{x^j r_1 e^{r_1x}} - \cancel{r_1x^j e^{r_1x}}$$

so $(D-r_1I)^{k_1+1}(x^j e^{r_1x}) =$

$$j(j-1)\cdots(j-k_1)\underbrace{x^{j-(k_1+1)}e^{r_1x}}_{\in V}$$

∎

<u>Example</u> (p. 4 Wed.)

let $L(y)=y''+2y'+y$.

Solve $L(y)=e^{-x}$

notice $p(r)=r^2+2r+1=(r+1)^2$

So you guess?

ans: $y=\frac{1}{2}x^2e^{-x}+c_1e^{-x}+c_2xe^{-x}$.

There's a method that's guaranteed to find a y_p for $L(y_p) = f$, if you've already found y_H – it looks magic now but will be motivated better later in course.

<u>variation of parameters</u> : If, for $L(y) := y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y$

you already have a basis for $\ker(L)$, i.e. the sol'n set to $L(y) = 0$, there is a formula to find a particular sol'n y_p to

$$L(y_p) = f.$$

(this will work for any right hand side, not just the special ones we considered for "undetermined coeff".)

derivation: Let $\{y_1, y_2, \dots y_n\}$ a basis for $\ker(L)$

we search for y_p expressed as

$u_j = u_j(x)$ fcns! (hence the name of the method).

$$\begin{array}{ll} a_0(& y_p(x) = u_1 y_1 + u_2 y_2 + \dots + u_n y_n \\ a_1(& y_p' = u_1 y_1' + u_2 y_2' + \dots + u_n y_n' + u_1' y_1 + u_2' y_2 + \dots + u_n' y_n \longrightarrow \text{set} \equiv 0 \\ + a_2(& y_p'' = u_1 y_1'' + u_2 y_2'' + \dots + u_n y_n'' + u_1' y_1' + u_2' y_2' + \dots + u_n' y_n' \longrightarrow \text{set} \equiv 0 \\ & \vdots \\ + a_{n-1}(& \\ + 1(& y_p^{(n)} = u_1 y_1^{(n)} + u_2 y_2^{(n)} + \dots + u_n y_n^{(n)} + u_1' y_1^{(n-1)} + \dots + u_n' y_n^{(n-1)} \longrightarrow \text{set} \equiv f \end{array}$$

$$\Longrightarrow \quad L(y_p) = \underbrace{u_1 L(y_1) + \dots + u_n L(y_n)}_{=0!} + f$$

conditions in matrix form

$$\underbrace{\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)} \end{bmatrix}}_{W(y_1, \dots y_n)} \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f \end{bmatrix}$$

$$\boxed{\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = [W]^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f \end{bmatrix}}$$

then find choices of each $u_j(x)$ by antidifferentiation,

& get $y_p = u_1 y_1 + \dots + u_n y_n$!

<u>Use Theorem</u>: since $\{y_1, \dots y_n\}$ is a basis we can solve all IVP's at each $x \Rightarrow [W]$ has rank n <u>at each</u> $x \Rightarrow W^{-1}$ exists at each x!

Example

forced spring, $mx'' + cx' + kx = F(t)$; e.g.

$$x''(t) + 16x(t) = \sin 4t$$

what would your guess be for $x_p(t)$ using guessing? ← work problem this way too!

$$L(x) := x'' + 16x$$

$$\ker L = \operatorname{span}\{\underset{\substack{\uparrow \\ x_1}}{\cos 4t}, \underset{\substack{\uparrow \\ x_2}}{\sin 4t}\}$$

var par formula from page 1:

$$\begin{bmatrix} \cos 4t & \sin 4t \\ -4\sin 4t & 4\cos 4t \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \sin 4t \end{bmatrix}.$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4\cos 4t & -\sin 4t \\ 4\sin 4t & \cos 4t \end{bmatrix} \begin{bmatrix} 0 \\ \sin 4t \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -\sin^2 4t \\ \sin 4t \cos 4t \end{bmatrix}$$

$$u_1' = -\frac{\sin^2 4t}{4} = \frac{\cos 8t - 1}{8} \qquad \text{(double/half angle formula)}$$

$$u_2' = \frac{1}{4}\sin 4t \cos 4t$$

choose $u_1 = \frac{\sin 8t}{64} - \frac{t}{8}$

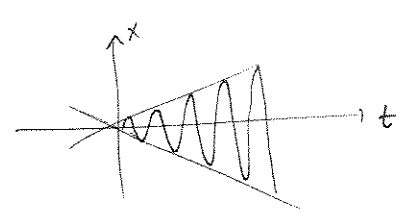
$u_2 = \frac{\sin^2 4t}{16}$

$$\cancel{y_p} \quad x_p = u_1 x_1 + u_2 x_2 = \left(\frac{\sin 8t}{64} - \frac{t}{8}\right)(\cos 4t) + \frac{\sin^2 4t}{32}(\sin 4t)$$

$$= \frac{2}{64}\sin 4t \cos 4t \cos 4t - \frac{t}{8}\cos 4t + \frac{1}{32}(1 - \cos^2 4t)(\sin 4t)$$

$$= -\frac{t}{8}\cos 4t + \underbrace{\frac{1}{32}\sin 4t}_{\text{solves } L(x) = 0}$$

so may take $\boxed{x_p(t) = -\frac{t}{8}\cos 4t}$



resonance!