

Math 2280-1
Wed Feb. 17

practice exams & solns
are posted!

(1)

§3.5 (not on Fri. exam.)

We are now experts at finding the general homogeneous soln to n^{th} order const. coeff DE's, i.e.

for

$$L = y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y \quad \text{with } a_j = \text{const}, j=0,1,\dots,n-1$$

We can always construct a basis for the n -dim'l space of solns to

$$L(y) = 0.$$

§3.5 contains algorithms for finding particular solns for the non-homogeneous DE, $L(y_p) = f$

(and then the full soln will be $y = y_p + y_H$)

Method 1 "Method of undetermined coeffs" (i.e. guessing)
(Actually depends on linear algebra)

example 1 : $y' + 4y = e^x$
 $y_H = Ce^{-4x}$

try $y_p = Ce^x$ since $L(e^{rx}) = p(r)e^{rx}$
so for $V = \text{span}\{e^x\}$,
 $L: V \rightarrow V$.

$$\begin{array}{l} 4 (y_p = Ce^x \\ 1 (y_p = Ce^x \end{array}$$

$$L(y_p) = 5Ce^x \stackrel{\text{want}}{=} e^x$$

$$C = 1/5$$

$$y = y_p + y_H = \frac{1}{5}e^x + Ce^{-4x}$$

example 2

$$y' + 4y = \cos 2x$$

need to include this fun!

for $V = \text{span}\{\cos 2x, \sin 2x\}$

$$L: V \rightarrow V$$

on V , $\ker L = \{0\}$ (since on $C^1(\mathbb{R})$, $\ker L = \text{span}\{e^{-4x}\}$)

so $\exists! v \in V$ s.t. $Lv = \cos 2x$ (rank + nullity thm!).

book's way.

4(try $y_p = A \cos 2x + B \sin 2x$

1($y_p' = -2A \sin 2x + 2B \cos 2x$

$$L(y_p) = \cos 2x [4A + 2B] + \sin 2x [-2A + 4B] \stackrel{\text{want}}{=} \cos 2x [1] + \sin 2x [0]$$

$$\begin{bmatrix} 4 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 4 & -2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{1}{10} \end{bmatrix}$$

$$y_p = \frac{1}{5} \cos 2x + \frac{1}{10} \sin 2x$$

clever 2270 graduate's way. (equivalent, but quicker!)

basis $\mathcal{B} = \{\overset{f_1}{\cos 2x}, \overset{f_2}{\sin 2x}\}$

matrix for L wrt $\mathcal{B}$ $\rightarrow [L]_{\mathcal{B}} = \left[\begin{array}{c|c} [L(f_1)]_{\mathcal{B}} & [L(f_2)]_{\mathcal{B}} \end{array} \right]$

$$= \begin{bmatrix} 4 & 2 \\ -2 & 4 \end{bmatrix}$$

$$\begin{array}{c} \uparrow \qquad \qquad \uparrow \\ L(\cos 2x) \qquad [L(\sin 2x)]_{\mathcal{B}} \\ = 4 \cos 2x - 2 \sin 2x \end{array}$$

so we want to solve this eqn for the coords of the sol'n!

example 3: find y_p for $L(y_p) = x + 2$

hint: let $V = P_1 = \text{span}\{1, x\}$.

$$\text{ans: } y_p = \frac{1}{4}x + \frac{7}{16}$$

Example

④ Using previous 3 examples, find the full sol'n to

$$y' + 4y = 2e^x - 3\cos 2x - 5x - 10$$

hint: superposition.

See guessing table page 205

f(x) (or a piece of f) e^{rx} $e^{ax} (A \cos bx + B \sin bx)$ $P_n(x)$ $P_m(x)e^{rx}$ etc

guess

 Ce^{rx} $e^{ax} (D \cos bx + E \sin bx)$ $Q_n(x)$ $Q_m(x)e^{rx}$ $V = \text{span}\{e^{rx}\}$ $L: V \rightarrow V$ L is 1-1 & onto iff r is not a root of charact poly. $V = \text{span}\{e^{ax} \cos bx, e^{ax} \sin bx\}$ L is 1-1 & onto iff $\pm bi$ not roots of char. poly $V = \text{span}\{1, x, \dots, x^n\}$ L is 1-1 & onto iff 0 is not a root of c.p. $V = \text{span}\{e^{rx}, xe^{rx}, \dots, x^m e^{rx}\}$ works if r is not a root of charact poly.Fix it recipe if L has non-trivial kernel for original V :if

at least one term in guess solves homogeneous eqn,
 multiply entire guess by x^s where s is the smallest
 counting # s.t. no term in new guess solves homogeneous DE !!
 i.e. - Can you give vector space proof of why this work, using L 's factorization ??

Example

$$L(y) = y'' + 2y' + y$$

solve

$$L(y) = e^{-x}$$

$$\text{notice } p(r) = (r^2 + 2r + 1) = (r+1)^2$$

$$\text{ans: } y_p = \frac{1}{2}x^2e^{-x}!$$

1st order ODE's. Chapters 1.1-1.5 (not 1.6, 2.4-2.6)
2.1-2.3

- Recognize and solve DE's and IVP's for

$$\frac{dy}{dx} = f(x) \quad \S 1.2 \quad \text{antidiff}$$

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \quad \S 1.4 \quad \text{separable}$$

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \S 1.5 \quad \text{linear}$$

- slope fields, $\exists!$ for IVP $\S 1.3, 2.2$
phase portraits.

geometric meaning of $\frac{dy}{dx} = f(x, y)$ for the graph $y = y(x)$

sketch slope fields using isoclines

$\exists!$ thm for IVP

autonomous DE's $\frac{dy}{dx} = f(y)$

equilibrium solns

phase portraits

stability

- applications & modeling

separable

exp growth & decay

Newton's law of cooling

Torricelli

logistic / doomsday ext / harvesting ($\S 2.1$)

acceleration with drag terms ($\S 2.3$)

linear

mainly mixing problems ($\S 1.5$)

also some of $\S 2.3$ models

if $P(x)$ is const, i.e.

$$y' + a_0 y = f$$

this theory connects to chapter 3 $y = y_p + y_H \dots$
use q, e^{rt} , etc.

Higher order linear DE's ($3.1-3.4$)

- Why $L(y) := y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y$ is called linear
- Why the general sol'n to $L(y) = f$ is $y = y_p + y_H$; more general superposition
- Dimension of sol'n space to $L(y) = 0$
(and its relationship to $\exists!$ thm for IVP)
- linear ind & dep of fns
Wronskian matrix & determinant
- Solving $L(y) = 0$ if L has constant coeffs a_i
 $\chi_r, p(r)$, real roots (distinct, repeated), complex roots (distinct & repeated)
using complex exponentials, Euler, cos & sin addition angle formulas

Mechanical vibrations

- pendulum & spring models
simple harmonic motion (amp, phase, ang. freq, freq, period, etc. ABC triangle)