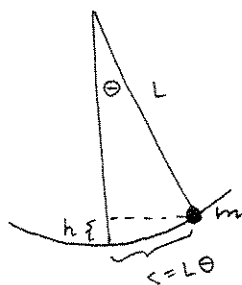


Math 2280-1
Tuesday Feb. 16

Experiment day! (and then finish discussing the harmonic oscillator with damping $\epsilon > 0$ ~ page 3 Friday notes ~ overdamped, underdamped, critically damped.)

pendulum



conservative system, $KE + PE = \text{const.}$

$$\frac{1}{2}mv^2 + mgh = \text{const}$$

$$s = L\theta$$

$$h = L - L\cos\theta = L(1 - \cos\theta)$$

$$v = \frac{ds}{dt} = L\theta'(t)$$

$$\text{so, } \frac{1}{2}mL^2(\theta')^2 + mgL(1 - \cos\theta) = \text{const}$$

$$D_{\theta}: mL^2\theta'\theta'' + mgL\sin\theta\theta' = 0$$

$$mL\theta' (L\theta'' + g\sin\theta) = 0$$

$\neq 0$, except
at isolated times.

~ deduce eqn of motion is

$$\theta'' + \frac{g}{L}\sin\theta = 0$$

linearize

$$\theta'' + \frac{g}{L}\theta = 0$$

$$\omega_0 = \sqrt{\frac{g}{L}}$$

$$\theta(t) = C\cos(\omega_0 t - \alpha)$$

non linear!

$$\text{but } \sin\theta = \theta - \frac{\theta^3}{3!} + \dots$$

$\sin\theta \approx \theta$ for θ small.

this is an excellent
approx.!

(alternating series test).

Prediction:

Here is my measurement of L , which we'll check again in class. It assumes the effective length of the pendulum uses the the distance to the ball's center of mass, from the top. (And, where exactly is the top?!)

```
> L:=1.528;
g:=9.806;
omega:=sqrt(g/L); #radians per second
f:=evalf(omega/(2*Pi)); #cycles per second
T:=1/f; #seconds per cycle
```

$$L := 1.528$$

$$g := 9.806$$

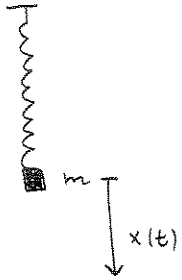
$$\omega := 2.533286258$$

$$f := 0.4031850301$$

$$T := 2.480250816$$

Experiment:

Error sources:

mass - spring

$$m x'' + k x = 0$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

prediction : (we'll check!)

Mass-spring:

How Hookes constant:

```
> 50.1-6.5; #add 50g and measure displacement
43.6
```

```
> m:=.1; #the mass is 100g = 0.1 kg
```

```
m:=0.1
```

My measurement for k, using a 50g mass and measuring the displacement in meters

```
> k*.436=.05*g;
0.436 k=0.49030
```

```
> k:=.05*g/.43;
k:=1.140232558
```

```
> omega:=sqrt(k/m);#radians per second
f:=omega/evalf(2*Pi); #cycles per second
T:=1/f; #seconds per cycle
```

```
omega:=3.376732974
```

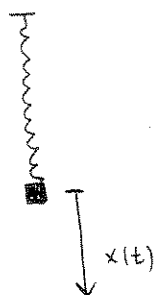
```
f:=0.5374237442
```

```
T:=1.860729100
```

experiment :

error sources ?
(there's a "very" big one!)

Improved spring model (account for spring mass)



normalize $KE + PE = 0$ for spring hanging at rest. (in gravity!)

then for system in motion

$$KE + PE = \frac{1}{2} m (\dot{x})^2 + KE_{\text{spring}} + \underbrace{\frac{1}{2} k x^2}_{\text{PE of system from equilibrium.}}$$

model: the top of the spring isn't moving. the bottom is moving with velocity $\dot{x}(t)$.

Assume the velocity of a piece of spring a fraction μ of the way ($0 \leq \mu \leq 1$) from top to bottom is $\mu \dot{x}(t)$.

so $\text{speed}^2 = \mu^2 (\dot{x})^2$, for that piece. Thus

$$KE = \int_0^1 \underbrace{\frac{1}{2} \mu^2 (\dot{x}(t))^2}_{v^2} \underbrace{m_s d\mu}_{\text{mass}} = \frac{1}{6} m_s (\dot{x}(t))^2$$

So for $M := m + \frac{1}{3} m_s$,

$$KE + PE = \frac{1}{2} M (\dot{x})^2 + \frac{1}{2} k x^2 \equiv \text{const}$$

$$D_t: \quad M \ddot{x} + k x \equiv 0$$

$$x' (\underbrace{M \ddot{x} + k x}_{=0}) \equiv 0$$

$$\omega_0 = \sqrt{\frac{k}{M}}$$

Does this improve our prediction?

```
> ms:=.011; #spring has mass 11g
M:=m+ms/3; #"effective mass"
M:=0.1036666667

> omegal:=sqrt(k/M); #new angular freq est.
f1:=omegal/evalf(2*Pi); #new freq est.
T1:=1/f1; #new period est
omega1:=3.316478236
f1:=0.5278339048
T1:=1.894535366
```

>

• Now discuss $m x'' + c x' + k x = 0$, $c > 0$; page 3 Friday notes.

The following examples may have also been relevant to our experiments

Example 1, page 189

$$m = \frac{1}{2} \text{ kg}$$

when $x = 2 \text{ m}$, force of spring is 100 N .

$$\text{i.e. } 2k = 100 \quad k = 50 \text{ N/m}$$

$$\frac{1}{2} x'' + 50 x = 0$$

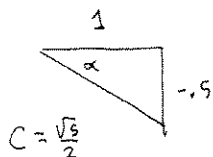
$$x'' + 100 x = 0 \quad \omega_0 = 10$$

$$\text{IVP } \begin{cases} x_0 = 1 \text{ m} \\ v_0 = -5 \text{ m/s.} \end{cases}$$

$$x(t) = A \cos 10t + B \sin 10t$$

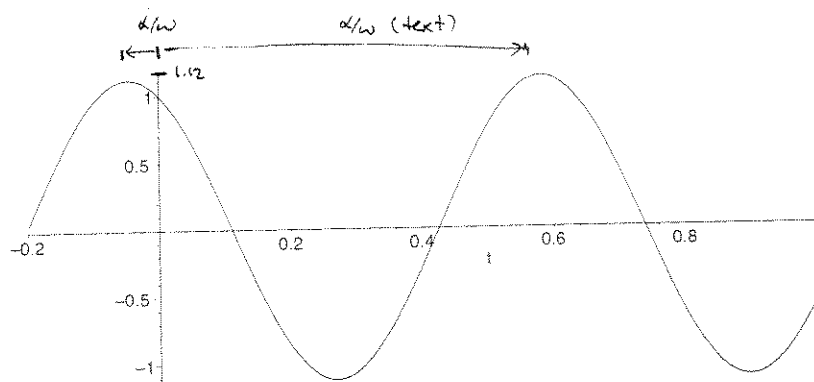
$$x(t) = \cos 10t - .5 \sin 10t$$

$$x(t) = \frac{\sqrt{5}}{2} \cos(10t - \alpha) \approx (1.12) \cos(10t + .46)$$



$$\alpha = \tan^{-1}(-.5) \\ \approx -.46 \text{ rad.}$$

(compare with text page 326,
says $\alpha = 5.8195$.
both answers are correct!)



Example 2, page 192

add a little damping ($c=1$) to example 1.

$$\frac{1}{2}x'' + x' + 50x = 0$$

$$\begin{cases} x'' + 2x' + 100x = 0 \\ x(0) = 1 \\ x'(0) = -5 \end{cases}$$

$$r^2 + 2r + 100 = 0$$

$$(r+1)^2 + 99 = 0$$

$$r = -1 \pm i\sqrt{99}$$

$$\begin{aligned} \text{or } r &= \frac{-2 \pm \sqrt{4 - 400}}{2} \\ &= -1 \pm i\sqrt{99} \end{aligned}$$

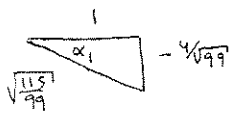
$$x(t) = e^{-t} (A \cos \sqrt{99}t + B \sin \sqrt{99}t) \quad \leftarrow \text{note, damping retards angular freq. } \omega_1 = \sqrt{99} \approx 9.95$$

$$x(0) = 1 = A$$

$$x'(0) = -5 = -A + \sqrt{99}B$$

$$B = -\frac{4}{\sqrt{99}}$$

$$\begin{aligned} x(t) &= e^{-t} \left[\cos(\sqrt{99}t) - \frac{4}{\sqrt{99}} \sin(\sqrt{99}t) \right] \\ &= e^{-t} \left[C_1 \cos(\sqrt{99}t - \alpha_1) \right] \approx \boxed{1.078 e^{-t} \cos(9.95t - 0.38)} \end{aligned}$$



α_1 larger

$$\alpha_1 = \tan^{-1}\left(-\frac{4}{\sqrt{99}}\right)$$

