Exam 1 next Friday 2/19
covers thru §3.4

Hw for Fri 2/19
§3.4 4, 5, 6 13, 15, 19, 23
plus study for exam!

Math 2280-1
Friday Feb. 12

We're considering constant coefficient, linear, homogeneous DE's

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

Are discussing algorithms for finding bases for the n-dim'l solution space, in terms of the characteristic polynomial

$$p(r) = r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0$$

and its roots.

We need to finish the discussion of what to do about complex roots,

- Wednesday notes

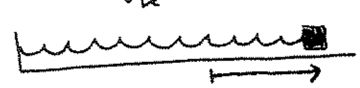
Then, begin §3.4: applications to mechanical oscillation problems

$y(x)$ ⟿ $x(t)$
theory — applications

Recall the linearized spring equation from earlier in the course:

$$m x'' + c x' + k x = f(t)$$

external driving force $\equiv 0$ in §3.4



$x(t)$ = displacement from equil.

$m x''$ = net forces $\approx -kx - \mu x' + f(t)$ linear approx

We study this extremely important DE in stages:

Case 1 Free undamped motion

no driving force in §3.4; $c = 0$

$$m x'' + k x = 0$$

$$x'' + \frac{k}{m} x = 0$$

$$x'' + \omega_0^2 x = 0 \qquad \left(\omega_0 := \sqrt{\tfrac{k}{m}}\right)$$

if $x = e^{rt}$, $p(r) = r^2 + \omega_0^2 = 0$

$r = \pm i \omega_0$

$e^{i\omega_0 t} = \cos \omega_0 t + i \sin \omega_0 t$

$$\boxed{x(t) = A \cos \omega_0 t + B \sin \omega_0 t \quad \text{simple harmonic motion}}$$

$= C \cos(\omega_0 t - \alpha)$!!

$= C \cos(\omega_0 (t - \delta))$

<u>example 1</u> A mass of 2 kg oscillates without damping on a spring with Hooke's constant 18 Newtons/meter. It is initially stretched 1 meter from equilibrium, and released with a velocity of $3/2$ m/sec. Show that the displacement $x(t)$ solves

A) $$\begin{cases} x'' + 9x = 0 \\ x(0) = 1 \\ x'(0) = 3/2 \end{cases}$$

B) Solve the IVP (A). Express the solution in amplitude-phase form, and interpret the $(t, x(t))$ graph below (Identify amplitude, phase, time delay).

```
> with(plots):
  Digits:=5:
> omega:=3;                 #angular frequency
  alpha:=arctan(.5);        #phase
  delta:=arctan(.5)/3;      #time delay
  C:=sqrt(5./4);            #amplitude
  T:=evalf(2*Pi/3);         #period
```

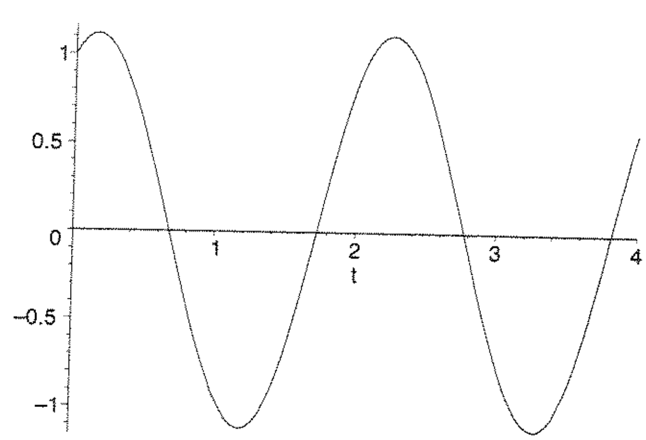
0.46365

0.15455

1.1180

$T := 2.0944$

```
> plot1:=plot(cos(3*t)+.5*sin(3*t),t=0..4,color=black):
  plot2:=plot(C*cos(3*t-alpha),t=0..4,color=black):
  plot3:=plot(C*cos(3*(t-delta)),t=0..4,color=black):
  display({plot1,plot2,plot3});
```



Case 2: Damping

$$m x'' + c x' + k x = 0$$

$$x'' + 2p x' + \omega_0^2 x = 0 \qquad \omega_0 = \sqrt{\tfrac{k}{m}} \text{ still}; \quad \tfrac{c}{m} = 2p; \quad p = \tfrac{c}{2m}$$

$$p(r) = r^2 + 2pr + \omega_0^2 = 0$$

$$r = \frac{-2p \pm \sqrt{4p^2 - 4\omega_0^2}}{2}$$

$$\boxed{r = -p \pm \sqrt{p^2 - \omega_0^2}}$$

Overdamped: $p^2 - \omega_0^2 > 0 \quad (c^2 > 4km)$

$$r = r_1 < r_2 < 0$$

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = e^{r_1 t}\left(c_1 + c_2 e^{(r_2 - r_1)t}\right)$$

Figure 3.4.7 p. 191

$\rightarrow$ sol's decay exponentially to zero, cross t-axis at most 1 time.

Critically damped $p^2 - \omega_0^2 = 0 \quad (c^2 = 4km)$

$r = -p$ double root

$$x(t) = c_1 e^{rt} + c_2 t e^{rt} = e^{rt}(c_1 + c_2 t)$$

Figure 3.4.8 p. 191 $\rightarrow$ sol's decay exponentially to zero, cross t-axis at most once

Underdamped $p^2 - \omega_0^2 < 0$

set $\omega_1 = \sqrt{\omega_0^2 - p^2} < \omega_0$

$$r = -p \pm i\omega_1$$

$$e^{rt} = e^{(-p \pm i\omega)t}$$

$$x(t) = e^{-pt}(A \cos \omega_1 t + B \sin \omega_1 t)$$

$$= e^{-pt}(C \cos(\omega_1 t - \alpha))$$

Figure 3.4.9. p. 192

1. solution oscillates, but osillations are damped exponentially
2. motion is slowed relative to no damping ($\omega_1 < \omega_0$).

pseudo period $T = \frac{2\pi}{\omega_1}$

pseudo freq $f = \frac{\omega_1}{2\pi}$

pseudo ang. freq ω_1

Ce^{-pt} : " time varying amplitude

Examples & experiments next week!