Math 2280-1
Monday Feb. 1

§2.3: improved velocity/acceleration models
(and §2.2: harvesting a logistic population)

In calc: (and §1.2)

$\uparrow y$ $\quad m\frac{dv}{dt} = F_G = -mg$

$$\frac{dv}{dt} = -g$$
$$v = -gt + v_0$$
$$y = -\tfrac{1}{2}gt^2 + v_0 t + y_0$$

add resistance?

$$m\frac{dv}{dt} = F_G + F_f$$

$$|F_f| \approxeq k|v|^p \qquad 1 \le p \le 2 \quad \text{empiricle}$$

$$\boxed{\begin{array}{l}\underline{p=1} \quad (\text{"linear" model}) \\ m\frac{dv}{dt} = -mg - kv\end{array}}$$

force will cause an acceleration opposite to direction of motion

... this is low speed "linearization"

if $F_f = F(v) = F(0) + F'(0)v + \frac{1}{2!}F''(0)v^2 + \dots$

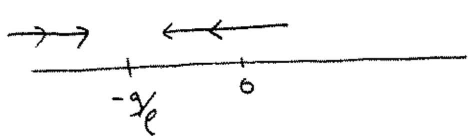
$F(0) = 0$

for $|v|$ small, "negligable"

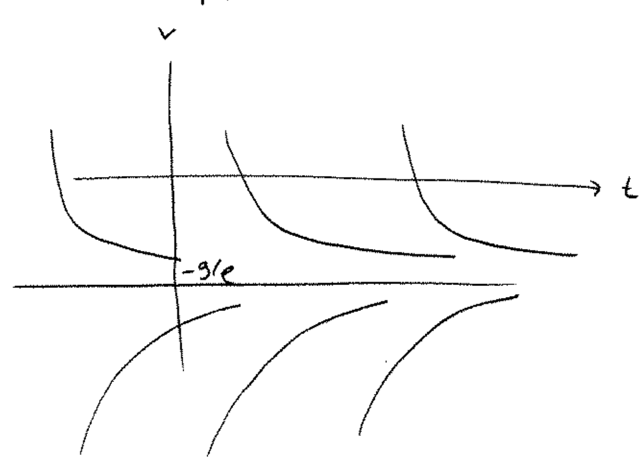
$$\frac{dv}{dt} = -g - \rho v \qquad \rho := \frac{k}{m}$$
$$= -\rho\left(v + \frac{g}{\rho}\right)$$

equil soltn: $v = -g/\rho$

phase portrait:



slope field:



analytic sol'n

$$\frac{dv}{dt} = -\rho\left(v + \frac{g}{\rho}\right)$$

$$\frac{dv}{v + g/\rho} = -\rho\, dt$$

$$\ln|v + g/\rho| = -\rho t + C \qquad t = 0 \Rightarrow C = \ln|v_0 + g/\rho|$$

$$= -\rho t + \ln|v_0 + g/\rho|$$

$$|v + g/\rho| = |v_0 + g/\rho| e^{-\rho t}$$

$$v + g/\rho = (v_0 + g/\rho) e^{-\rho t} \qquad \text{(why?)}$$

$$\boxed{v = \underbrace{-g/\rho}_{v_T} + (v_0 + g/\rho) e^{-\rho t}}$$

"terminal velocity"

$$y = \int v(t)\,dt = t v_T + \left(\frac{v_0 - v_T}{-\rho}\right) e^{-\rho t} + C$$

$$y_0 = \frac{v_T - v_0}{\rho} + C$$

so $C = y_0 + \frac{v_0 - v_T}{\rho}$

$$\boxed{y = t v_T + \left(\frac{v_0 - v_T}{\rho}\right)(1 - e^{-\rho t}) + y_0}$$

Examples 1 & 2 p. 98-100

crossbow bolt

$v_0 = 49$ m/s

$g = 9.8$ m/s^2

$y_0 = 0$

no friction

$y = -4.9 t^2 + 49 t$

$v = -9.8 t + 49$

max ht at $t = 5$ sec

$y_{max} = y(5) = 49(2.5)$

$= 122.5$ m

time aloft = 10 sec.

linear drag

$\rho = .04$ (drag coeff; empirical)

corresponds to

$|v_T| = \frac{g}{\rho} = \frac{9.8}{.04} = 245$ m/sec

(you could measure this to deduce ρ)

$v_0 - v_T = 49 + 245 = 294$

so

$v = -245 + 294 e^{-.04t}$

$v = 0$ at $\frac{245}{294} = e^{-.04t}$

$t_{max} \approx 4.56$ sec

$y = -245 t + (294)(25)(1 - e^{-.04t})$

$y(t_{max}) = ?$

When does bolt hit ground?

Computation sheet for Example 2, section 2.3

Time of maximum height:

```
> 25*ln(294.0/245);
  solve(-245+294*exp(-.04*t)=0,t); #should be same
```

$$4.558038920$$

$$4.558038920$$

Formulas for height and velocity functions

```
> v:= t -> 294*exp(-.04*t) - 245;
```

$$v := t \to 294\, e^{(-0.04\, t)} - 245$$

```
> y:= t-> -245*t + 294*25*(1 - exp(-.04*t));
```

$$y := t \to -245\, t + 7350 - 7350\, e^{(-0.04\, t)}$$

Maximum height, return time tor ground time spent falling, landing speed:

```
> y(4.558038920); #max height
```

$$108.280465$$

```
> solve(y(t)=0,t);  #find when returns to ground
```

$$9.410949931, 0.$$

```
> 9.410949931 - 4.558038920; #time descending
```

$$4.852911011$$

```
> v(9.410949931); #speed when it lands
```

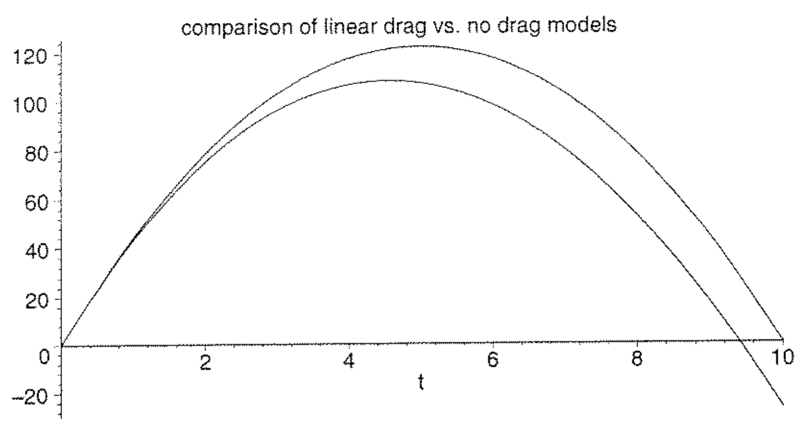
$$-43.2273093$$

Conclusions: bolt rises for 4.56 seconds, to a height of 108.3 meters. Then it spends 4.85 seconds descending, landing with a velocity of -43.3 meters per second.

```
> with(plots):
Warning, the name changecoords has been redefined
> z:= t->-4.9*t^2 + 49*t;  #the no resistance model
```

$$z := t \to -4.9\, t^2 + 49\, t$$

```
> plot({z(t),y(t)}, t = 0..10, color=black,
  title='comparison of linear drag vs. no drag models');
```



quadratic drag is also interesting...

going up:

$$m\frac{dv}{dt} = -mg - kv^2$$

$$\frac{dv}{dt} = -g\left(1+\frac{k}{g}v^2\right)$$

$$\frac{dv}{1+\frac{k}{g}v^2} = -g\,dt \;..$$

arctan!

going down:

$$m\frac{dv}{dt} = -mg + kv^2$$

$$\frac{dv}{dt} = -g\left(1-\frac{k}{g}v^2\right)$$

$$\frac{dv}{1-\frac{k}{g}v^2} = \cancel{d} -g\,dt$$

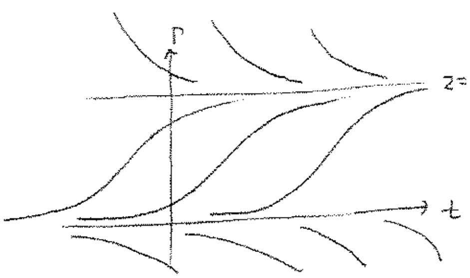
parfrac! (or $\tanh^{-1}$)

§2.2 backtrack, if time:
Now, let's harvest a logistic population! (e.g. fisheries).

$$\frac{dP}{dt} = \underbrace{aP - bP^2}_{\text{logistic}} \; \underbrace{-\, h}_{\text{constant rate harvesting.}}$$

(see §2.2 #24)
one could also consider a term $-hP$, which would maybe correspond to constant effort harvesting (see §2.2 #23)

e.g. $\frac{dP}{dt} = 2P - P^2$
$= P(2-P)$



vs $\frac{dP}{dt} = 2P - P^2 - h$

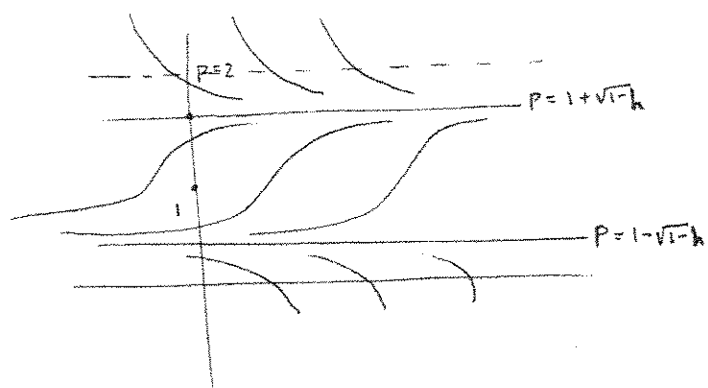
consider what happens for different h values

roots of RHS: $P^2 - 2P + h = 0$ has roots
$$P = \frac{2 \pm \sqrt{4-4h}}{2} = 1 \pm \sqrt{1-h}$$

so, for $0 \le h \le 1$,
$$\frac{dP}{dt} = -\left(P - (1+\sqrt{1-h})\right)\left(P - (1-\sqrt{1-h})\right)$$

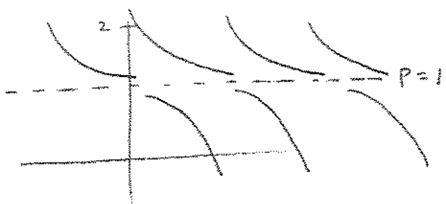
still looks logistic (& fishery won't collapse) if $h \ll 1$

extinction zone

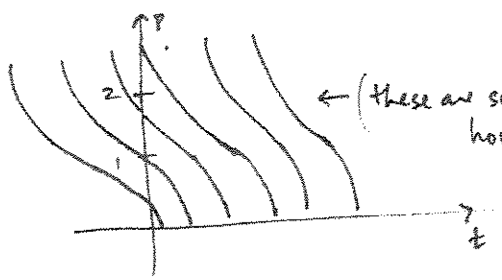


as $h \nearrow 1^-$ the middle zone (where populations increase to the modified carrying capacity) shrinks in width, until at $h=1$ the two roots coalesce;

$$\frac{dP}{dt} = 2P - P^2 - 1 = -(P^2 - 2P + 1) = -(P-1)^2$$



and for $h > 1$, $\quad 2P - P^2 - h = -(P^2 - 2P + h) = -((P-1)^2 + h - 1) < 0 \quad \forall P \quad$ (but least negative) @ $P = 1$

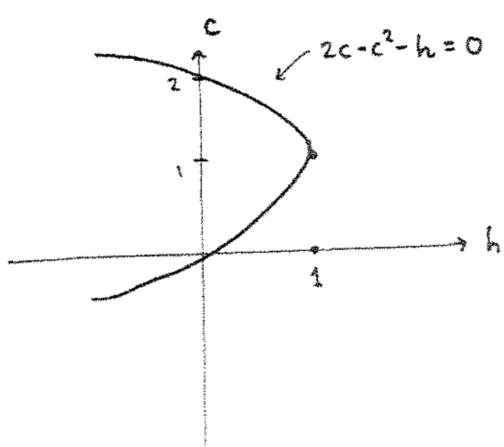


(these are supposedly horizontal translations of each other)

— Extinction $\forall P_0$!!

this model gives a plausible explanation for why fishery after fishery has collapsed around the world – if $h < 1$ but near 1, and if something perturbs the system a little bit (e.g. increased fishing pressure, a big storm, etc.), you could be confronted with "sudden", unexpected fishery collapse.

"bifurcation diagram" of equilibrium sol'ns, in the h-c plane:



$h < 0$ stocking!

$h > 0$ harvesting